SUSCEPTIBILITY OF BINARY DPSK
TO PERIODIC FM INTERFERENCE

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Cecil A. Lowery, Jr.

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SUSCEPTIBILITY OF BINARY DPSK

TO PERIODIC FM INTERFERENCE

Approved:

R. H. Pettit, Chairman

J. E. Brown

T. P. Barnwell

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SUMMARY

In recent years, the design of electronic communication systems has placed an increasing reliance on the use of electromagnetic radiation as the connecting link between system elements. As a result, the likelihood of either deliberate or unintentional man-made interference of these systems has increased accordingly. In what is considered to be the normal mode of operation, these systems are perturbed only by natural phenomena, such as thermal and atmospheric noise, fading, and multipath transmissions, the analyses for which have been numerous and well documented in the literature.

The analysis described herein involves the use of theoretical and computational techniques in a study of the performance of binary differential-phase-shift-keyed (DPSK) communication systems operating in the presence of both Gaussian noise and intentional interference from a periodically modulated FM jamming source. A determination is made of DPSK system susceptibility in terms of the probability of receiver error in the decision as to which binary digit is transmitted.

An involved theoretical equation, relating the system error to the operating characteristics of the system, noise, and interference, is first derived for the case of general periodic FM interference. Simplification is obtained by consideration of certain classes of the general interference. These classes contain the important types of interfering signals, e.g., cochannel and interchannel interferers

operating in spot, swept-spot, or barrage jamming modes with arbitrary received power levels. In addition, the analysis provides for a variation of system signal-to-noise ratios and for different bandpass filter implementations in the DPSK receiver. The receiver bandpass filters considered are of the Butterworth, Chebyshev, and Ideal types, of varying order.

Numerical computation methods are described and used for obtaining quantitative results. Modulation types chosen for quantitative analysis are linear-swept FM, FM by sinusoidal tone, and CW in the limiting case of zero modulation, each operating in either cochannel or adjacent channel modes. Of these types, it has been established that the DPSK system is more susceptible to linear FM interference than to sinusoidal FM. For the cochannel interferences, CW interference is superior to either of the other modulated forms with one important exception, that of linear FM with low modulation index. When the system employs high-order bandpass filtering, this type of interference can induce higher error rates in the system than CW with the same interfering power. For the adjacent channel interferences, high-order bandpass filtering decreases the error rate from that obtained when employing first-order types. Furthermore, no significant difference in system error rate has been found to exist between employing low ripple Chebyshev or Butterworth filters of identical order.

For reference, a number of error rate families are presented

for first-order Butterworth filtering of unity modulation index, cochannel interference. Degradations from these curves for other interfering cases are presented and graphical techniques are explained for generating similar error rate families for these cases.

CHAPTER I

INTRODUCTION

Electronic jamming may be generally defined as the interaction of two or more communication systems for the purpose of intentional interference with the proper operation of one or more of these systems. In this context, a communication system is considered to be any electronic device which transmits and/or receives information. The following problems are of fundamental importance in any analysis of a jamming environment:

1. When, where, and how to generate the interference (electronic counter-measure).
2. Determination of any resulting detrimental effects on the system.
3. How to counter any detrimental effects (electronic counter-counter measures).

The areas in which electronic counter-measures (ECM) have found the widest application are in the jamming of enemy radar, communication, and missile-guidance systems, primarily with respect to those associated with unfriendly weapons systems. Particular interest has been paid toward jamming those weapons systems capable of destroying the vehicle or location of the ECM equipment, e.g., aircraft jamming of enemy radar.

In recent years, the design of weapons systems has placed more and more reliance on the use of electromagnetic radiation as a connecting link between system elements. The inherent weakness in this approach is that the radiation of RF signals is not generally secure from unfriendly detection and interference. Considerable effort may be expended to make the communication links of weapons systems secure, but if sufficient information can be obtained as to the transmission characteristics of these links, security will be increasingly difficult to maintain.

The question of security is of fundamental importance in applying communication systems in the presence of enemy ECM techniques. Basic to this question are the following characteristics of the system:

(a) Interceptibility: This is a measure of the ease with which the enemy can electronically intercept and/or identify the presence and location of the weapons system. Interceptibility includes the determination of intercept receiver sensitivity required to intercept and identify the RF transmission characteristics versus range.

(b) Accessibility: In order for the enemy to take advantage of any susceptible points in a system, he must be able to reach these points with a jamming counter-measure of appropriate form and sufficient power. Accessibility includes determination of jammer power at the receiving station, considering such factors as jammer location and output power, antenna lobe patterns, propagation conditions, etc.

(c) Susceptibility: This is a measure of the design of the system, which determines the effect of enemy ECM on its performance.

Susceptibility includes the examination of system performance in the presence of any type of enemy ECM, and its influence on effective system performance. The degradation due to specific jamming types and parameters (power bandwidth, modulation, deviation, pulse rate, pulse width, etc.) must be evaluated.

Among the possible types of jamming, the most likely to be encountered are the following:

(1) Repeater Jamming

(2) Spot-frequency CW

(3) Simultaneous spot frequencies

(4) Swept CW (linear sweep FM)

(5) Gaussian Noise

(6) AM & FM by sinusoidal tone(s)

(7) AM & FM by noise

(8) AM & FM by noise babble

(9) Pulse modulated CW, especially simulated radar signals.

Depending on their power bandwidths, the above types may be considered to fall into one of three modes:

(1) Spot-jamming mode. The power bandwidth is approximately the same as the receiver bandwidth. This allows insertion of high interfering power density into the receiver. The disadvantage of this method is the possible switching of the system frequencies to a new channel within the operating band (ECCM).

(2) Barrage jamming mode. The power is spread over the entire operating band of the receiver. This mode requires a significant

increase in jammer output power to maintain the power density of the spot jamming mode.

(3) Swept-spot mode. The intermediate case wherein the jamming frequency is swept across the complete operating band, allowing the high-power density of the spot jammer to appear for short periods of time on all the possible receiver channels.

Among these many possible types of jamming techniques, the use of frequency modulated (FM) interference provides the jamming source with a number of parameters by which the performance of a communications system can be affected, e.g., jammer power, carrier frequency, modulation (sweep) frequency, and sweep deviation. By suitable variation of these parameters, an FM jamming source can produce all three modes of jamming discussed previously.

In a digital communication system, the encoder selects one symbol of a symbol set for transmission during each signal interval. The receiver makes an observation of the received waveform during each signaling interval and decides which one of the symbols was selected. The decision making process is complicated by the presence of random noise and interfering signals which corrupt the transmitted waveforms, at times resulting in decision errors. The criterion of performance for these systems is the susceptibility of the system to decision error. The measure of performance is generally expressed in terms of the average probability of decision error.

The purpose of this research is to determine the error susceptibility of binary differential-phase-shift-keyed (DPSK) receivers [1] [2] when subjected to non-Gaussian interference resulting from the presence of unfriendly jamming. In general, the type of interference considered is constant amplitude frequency modulation with periodic modulating signal. Particular types of interference compared include linear-swept FM, sinusoidally-swept FM, and CW (zero-sweep).

The advent of DPSK for data transmission was first proposed by M. L. Doelz [3] and introduced by the development of the Collins "Kineplex" system [4] [5]. Error analysis of DPSK systems followed rapidly, when Lawton [6] and Fleck [7] formulated the result that the bit error probability for binary DPSK in the presence of additive white Gaussian noise is

$$\Pr(\text{err}) = \frac{1}{2}\exp(-E/N_o) \tag{1-1}$$

where

E is the energy per received symbol

N_o is the noise power spectral density (single-sided)

Furthermore, it was shown [6] that the minimum obtainable error rate of a binary data system operating in this environment is

$$\Pr(\text{err})_{\min} = \frac{1}{2}\left(1-\text{erf}\left(\sqrt{E/N_o}\right)\right) \tag{1-2}$$

where

$$\text{erf}(x) = (2/\sqrt{\pi}) \int_o^x \exp(-t^2)\,dt$$

and that the optimum system is a phase-shift keyed (PSK) system operating with perfect coherent detection.

Comparison of (1-1) and (1-2) for high E/N_o ratios ($\geq$10db) also showed that the disadvantage of binary DPSK with respect to optimum PSK is less than 1 db. As a result, it became clear that for high E/N_o ratios, binary DPSK performed almost as well as binary PSK, without the necessity of maintaining coherence in a local oscillator. More general analyses followed [8] [9]; however, these analyses were limited to reception of message and additive white noise only, whereas the increasing likelihood of the usage of DPSK systems in commercial, space, and military applications, generated questions as to the performance of DPSK in the presence of various forms of interference and/or operations requiring long-distance radio communications. Rosenbaum [10] analyzed the situation in which a single, constant amplitude, in-band, and additive interference is also included in the received signal. Jones [11] [12] investigated the effects on DPSK error rate of radio transmission over long distances involving fading multipath reflections, and demonstrated how the analysis could incorporate the case of pure CW tone and Gaussian noise [13].

The major drawback of the analyses to this point was the inclusion of only one interfering sinusoidal source, plus possibly multipath effects, into the Gaussian noise and signal environment. Rosenbaum [14] [15] further expanded the analysis to include the effects of multiple randomly-phased sinusoidal interferences which are cochannel with the message (same frequency as message carrier). In this case, the results are dependent on the assumption that each phase of each interfering sinusoid is indeed independent of others and uniformly distributed, i.e., the interfering sinusoids are independent and all cochannel with the message. Goldman [16] extended Rosenbaum's analysis to include multiple-error performance, with the additional restriction that the interference be independent from one signaling interval to the next.

The intentional jamming of a DPSK system by an unfriendly source using periodically modulated FM can, with suitable structuring, be represented by means of Fourier Series expansions [17] [18] [19], i.e., this type of interference consists of an infinite sum of sinusoids appearing at different frequencies and with specific (non-random) phase relationships with respect to each other. Due to this complex, but related, structure of FM interference, the previously mentioned techniques of analysis requiring independently phased sinusoids and/or cochannel frequencies are not applicable to the problem. In a recent analysis by Pettit [20] a non-coherent frequency-shift-keyed (NCFSK) system is analyzed for error rate in the presence of both white Gaussian noise and a linear-swept FM jamming signal. Theoretical results, including an equation for average error, were obtained for this system, however, no quantitative results are yet available.

The research reported herein extends the knowledge to DPSK systems operating in the presence of Gaussian noise, generalizes the interference to FM with periodic modulation, and obtains quantitative results. For purposes of analysis it is assumed that the DPSK system is both interceptible and accessible. It is thus implied that an unfriendly source is aware of the operating characteristics of the DPSK system, and has sufficient power to reach the system with an interfering FM signal.

CHAPTER II

THE SYSTEM MODEL

The DPSK Receiver Model

The analytical model of the DPSK receiver [1][2] is illustrated in Figure 1. The received signal r(t) consists of the sum of the DPSK signal $s_i(t)$, the jamming interference j(t), and white Gaussian noise n(t), i.e.,

$$r(t) = s_i(t) + j(t) + n_i(t) \tag{2-1}$$

where the subscript denotes the form of the data signal and noise component in the i^{th} signalling interval.

A narrow bandpass filter (BPF), centered at the signal carrier frequency, filters the received input to yield an output

$$d_i(t) = s_i(t) + j_f(t) + n_f(t) \tag{2-2}$$

in which:

(a) the BPF is assumed to pass the signal without distortion.

(b) $j_f(t)$ is the filtered version of j(t), since j(t) is generally wideband with respect to the BPF.

(c) $n_f(t)$ is a narrowband Gaussian noise process generated by the filtering process.

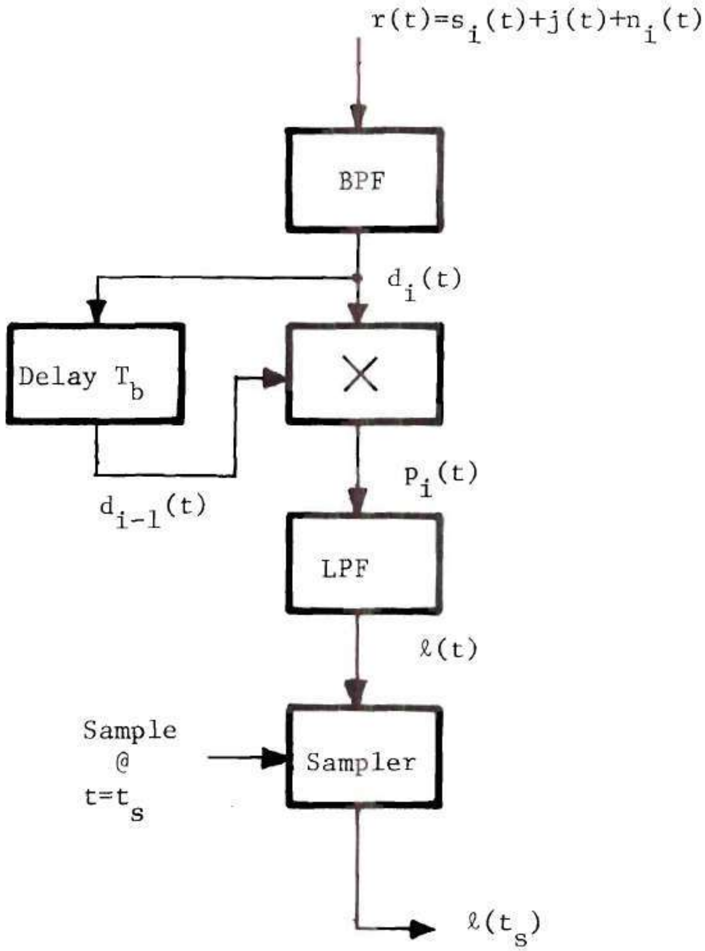

Figure 1. The DPSK Receiver

The filter output is then multiplied by the output $d_{i-1}(t)$ occurring during the previous interval, which is obtained by delaying the input for one bit period T_b. For maximum signal-to-noise ratio, T_b is an integral multiple of the carrier period T_c [2], i.e.,

$$T_b = nT_c \qquad n \text{ an integer} \tag{2-3}$$

or equivalently in terms of the carrier frequency w_c:

$$w_c T_b = 2\pi n \tag{2-4}$$

The output product $p_i(t)$ is then filtered by a lowpass filter (LPF) so as to remove any unwanted high frequency components generated in the multiplication process. The resulting output $\ell(t)$ is then sampled at time t_s, generating a random variable $\ell(t_s)$ which is compared to a zero threshold for the decision process. Thus if

$$\ell(t_s)>0 \quad : \quad \text{decide } H_1 \text{ true (1 sent)} \tag{2-5}$$

$$\ell(t_s)<0 \quad : \quad \text{decide } H_o \text{ true (0 sent)} \tag{2-6}$$

The probability of error for the system is

$$\Pr(\text{err}) = \Pr(1 \text{ sent}) \cdot \Pr(\text{decide } 0|1 \text{ sent}) \tag{2-7}$$

$$+ \Pr(0 \text{ sent}) \cdot \Pr(\text{decide } 1|0 \text{ sent})$$

or, in terms of the random variable $\ell(t_s)$:

$$\Pr(\text{err}) = P_1 \cdot \Pr[\ell(t_s)<0|H_1 \text{ true}] \tag{2-8}$$

$$+ P_o \cdot \Pr[\ell(t_s)>0|H_o \text{ true}]$$

Assuming transmission of "zeros" or "ones" is equally likely,

$$\Pr(\text{err}) = \frac{1}{2}\{\Pr[\ell(t_s)<0|H_1] + \Pr[\ell(t_s)>0|H_o]\} \tag{2-9}$$

The fundamental problem is to determine the probabilities in equation (2-9). These probabilities must be dependent in some manner on the signal, noise, interference, and receiver parameters.

The Signal Model

The binary DPSK signal is expressible as

$$s_i(t) = A_s \cos(w_c t + \theta_i) \qquad \theta_i = 0,\pi \tag{2-10}$$

in which A_s is the carrier amplitude, w_c the carrier frequency, and θ_i the phase of the i^{th} bit. The desired information is carried in the difference of phase between the i^{th} bit and the preceding bit, thus the designation "differential-phase-shift keying."

The Noise Model

The narrowband noise process $n_f(t)$ resulting from narrowband filtering of a white Gaussian noise process is expressible as [21]

$$n_f(t) = x_i(t) \cos w_c t - y_i(t) \sin w_c t \qquad (2\text{-}11)$$

in which $x_i(t)$ and $y_i(t)$ are quadrature noise components, each Gaussian distributed with zero mean, variance N, and independent of each other. Furthermore, it is assumed in the analysis that the narrowband noise autocorrelation vanishes at the bit interval delay T_b, i.e., x_i, x_{i-1}, y_i, y_{i-1}, are uncorrelated and hence independent [22]. The assumption of non-correlation, which allows for reasonable mathematics, is not unreasonable because the degree of correlation is negligible for the cases of interest [10], [14], [15], [16], [23]. The probability density function of the quadrature components at any time instant is thus the Gaussian density function [22]:

$$p(n) = (1/\sqrt{2\pi N}) \exp (-n^2/2N) \qquad (2\text{-}12)$$

The Bandpass Filter Model

The narrowband filter of Figure 1 is assumed to be a realizable causal filter (or the composite of more than one similar type). The filter is assumed symmetric about the signal carrier w_c. Thus if the filter response H(w) is defined as

$$H(w) = A(w) \exp[-j\theta(w)] \tag{2-13}$$

then [24]:

$$A(w_c + w) = A(w_c - w) \tag{2-14}$$

$$\theta(w_c + w) = -\theta(w_c - w) \tag{2-15}$$

where $A(w)$ and $\theta(w)$ are the amplitude and phase characteristics of the filter.

The FM Interference Model

The received FM interference $j(t)$ is defined as

$$j(t) = A_j \cos(w_j t + \phi_j + \int_o^t m(t_1)dt_1) \tag{2-16}$$

in which:

(a) A_j is the interfering carrier amplitude

(b) w_j is the interfering carrier frequency

(c) ϕ_j is a uniformly distributed random phase

(d) $m(t)$ is a periodic modulating waveform with period T_m, which varies the instantaneous frequency of the interference periodically about w_j with maximum deviation of Δw.

In addition, the beginning of the jammer sweep cannot be assumed to coincide with the start of the signaling interval. This complication is included by inserting an offset parameter t_o in the definition of $m(t)$. Figure 2 illustrates these points for the case of periodic linear-sweep and sinusoidal modulation of the FM interference.

Although equation (2-16) is sufficient to fully characterize the input interference, the filtered interference $j_f(t)$ is a quite complex function of the modulation $m(t)$ and the filter characteristics. A sufficient model of the filtered interference is obtained by first noting that since the modulation is periodic, its effect can be determined via a Fourier Series expansion. Equation (2-16) is first expressed as [17] [18] [19]:

$$j(t) = A_j R_e\{\exp[j(w_j t + \phi_j)]\exp[j\int_o^t m(t_1)dt_1]\} \qquad (2\text{-}17)$$

where $R_e\{\ \}$ denotes the "Real part of" operator. With $m(t)$ periodic,

$$\exp[j\int_o^t m(t_1)dt_1] = \sum_{n=-\infty}^{\infty} a_n \exp[jnw_m t] \qquad (2\text{-}18)$$

i.e., a complex Fourier Series expansion is used. The fundamental frequency is

$$w_m = 2\pi/T_m \qquad (2\text{-}19)$$

and the complex coefficients

$$a_n = (1/T_m)\int_o^{T_m}\exp[j\int_o^t m(t_1)dt_1]\exp[-jnw_m t]dt \qquad (2\text{-}20)$$

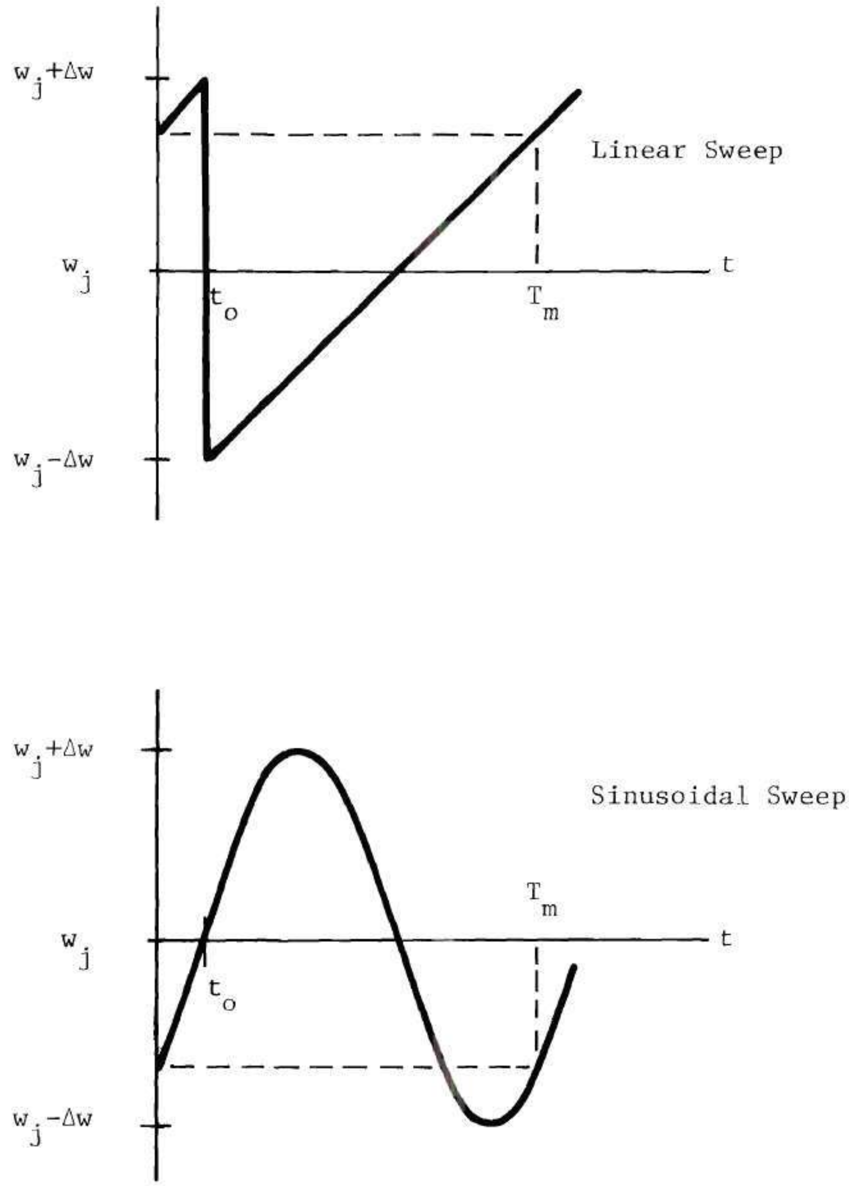

Figure 2. Modulation Waveforms

These coefficients must be evaluated for each type of modulation m(t), using equation (2-20). Since the coefficients are complex, an alternate expression is

$$a_n = |a_n| \exp [j\phi_n] \tag{2-21}$$

where $|a_n|$ is the magnitude of a_n and ϕ_n is the phase angle. Furthermore,

$$|a_n| = [a_n a_n*]^{1/2} \tag{2-22}$$

where a_n* is the conjugate of a_n, and

$$\phi_n = \arctan[I_m\{a_n\}/R_e\{a_n\}] \tag{2-23}$$

Substituting these definitions, equation (2-17) becomes

$$j(t) = A_j R_e\{\exp[j(w_j t + \phi_j)] \sum_{n=-\infty}^{\infty} |a_n| \exp[j(nw_m t + \phi_n)]\} \tag{2-24}$$

whence upon taking the indicated real part:

$$j(t) = A_j \sum_{n=-\infty}^{\infty} |a_n| \cos[(w_j + nw_m)t + \phi_j + \phi_n] \tag{2-25}$$

Equation (2-25) demonstrates that the FM interference is representable as an infinite sum of sinusoidal components centered about w_j, with amplitude, frequency, and relative phase of the components determined by the modulation m(t). The determination of the a_n coefficients for a modulation case of interest is developed in Appendix I.

Since the input interference consists of discrete sinusoidal components, the filtered version $j_f(t)$ can be obtained by modifying each input component by the filter amplitude and phase characteristics at the frequencies of interest. Thus:

$$j_f(t) = A_j \sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m) \cos[(w_j + nw_m)t + \phi_j + \phi_n - \theta(w_j + nw_m)] \tag{2-26}$$

While equation (2-26) specifies completely the filtered interference, the more reasonable approach for computational purposes is a truncation of the indicated infinite series when all the significant components (99% power level) are included in the sum.

CHAPTER III

THE SYSTEM SUSCEPTIBILITY

In this section, the relationships defining the system susceptibility, i.e., the equations for the probability of decision error, will be determined. A quite general but complex relationship will be shown to evolve, from which simpler relationships are often obtained when considering specific types of FM interferences. The approach to be taken consists first of expressing the filtered interference in a more suitable form, defining equivalent decision rules, and determining the error relationships of the system from these equivalent rules.

Envelope and Phase of the Filtered Interference

In the preceding chapter, it has been shown that the components of signal and noise are expressible in terms of quadrature components of the signal carrier frequency. In order to obtain a quadrature representation for the interfering component, the parameter

$$\rho_n(t) = (w_j - w_c + nw_m)t + \phi_n - \theta(w_j + nw_m) \tag{3-1}$$

is first defined. Using this definition, equation (2-26) may then be written as

$$j_f(t) = A_j \sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m) \cos[w_c t + \phi_j + \rho_n(t)] \tag{3-2}$$

whence, upon expanding the cosine argument:

$$j_f(t) = \{A_j \sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m)\cos[\rho_n(t)]\}\cos(w_c t + \phi_j) \quad (3\text{-}3)$$

$$- \{A_j \sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m)\sin[\rho_n(t)]\}\sin(w_c t + \phi_j)$$

The filtered interference can now be expressed in the form

$$j_f(t) = A_{jf}(t)\cos(w_c t + \phi_j + \phi_{jf}(t)) \quad (3\text{-}4)$$

where

$$A_{jf}(t) = A_j\{[\sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m)\cos[\rho_n(t)]]^2 \quad (3\text{-}5)$$

$$+ [\sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m)\sin[\rho_n(t)]]^2\}^{1/2}$$

and

$$\phi_{jf}(t) = \arctan \frac{\sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m)\sin[\rho_n(t)]}{\sum_{n=-\infty}^{\infty} |a_n| A(w_j + nw_m)\cos[\rho_n(t)]} \quad (3\text{-}6)$$

are the respective time varying envelope and phase components of the interference, with respect to the signal carrier.

An Equivalent Decision Rule

Equations (2-5) and (2-6) define the receiver threshold decision rule as used in a conventional binary DPSK system. To determine the error probability of (2-9) it is often first necessary to determine the random variable $\ell(t_s)$ for the system, derive the conditional probability density functions of $\ell(t_s)$ for each hypothesis and calculate the associated probabilities of error as given in (2-9). However, in many cases it is often more convenient to determine a mathematically equivalent decision rule from which to determine the error probability. The latter approach is chosen here.

Using equations (3-4) and (2-11) for the interfering and noise components, the filter output in the i^{th} bit interval is

$$d_i(t) = s_i(t) + A_{jf}(t)\cos[w_c t + \phi_j + \phi_{jf}(t)] \tag{3-7}$$

$$+ x_i(t)\cos(w_c t) - y_i(t)\sin(w_c t)$$

The remaining input, $d_{i-1}(t)$, to the multiplier of Figure 1 is the output $d_i(t)$ in the preceding bit interval delayed by the bit duration time T_b, i.e.,

$$d_{i-1}(t) = d_i(t - T_b) \tag{3-8}$$

Thus via equation (3-7) the delayed version is

$$d_{i-1}(t) = s_{i-1}(t) + A_{jf}(t - T_b)\cos[w_c(t - T_b) + \phi_j + \phi_{jf}(t - T_b)] \quad (3\text{-}9)$$

$$+ x_i(t - T_b)\cos(w_c t) - y_i(t - T_b)\sin(w_c t)$$

Assuming arbitrarily that $\theta_i = 0$ for $s_i(t)$ in the signal definition (2-10), then

$$s_i(t) = A_s \cos(w_c t) \quad (3\text{-}10)$$

$$s_{i-1}(t) = \pm A_s \cos(w_c t) \quad (3\text{-}11)$$

The choice of sign in (3-11) depends only on whether a phase change in the signal occurs in the adjacent signaling intervals.

Expanding the cosine arguments of (3-7) and (3-9), substituting (3-10) and (3-11), and noting that, via equation (2-4), the $w_c T_b$ term may be eliminated, the inputs can be written in a final quadrature form:

$$d_i(t) = \{A_s + A_{jf}(t)\cos[\phi_j + \phi_{jf}(t)] + x_i(t)\}\cos(w_c t) \quad (3\text{-}12)$$

$$- \{A_{jf}(t)\sin[\phi_j + \phi_{jf}(t)] + y_i(t)\}\sin(w_c t)$$

$$d_{i-1}(t) = \{\pm A_s + A_{jf}(t - T_b)\cos[\phi_j + \phi_{jf}(t - T_b)] + x_i(t - T_b)\} \quad (3\text{-}13)$$

$$\times \cos(w_c t) - \{A_{jf}(t - T_b)\sin[\phi_j + \phi_{jf}(t - T_b)]$$

$$+ y_i(t - T_b)\}\sin(w_c t)$$

For notational simplicity in the following, these definitions are made:

$$P_1 = A_s + A_{jf}(t)\cos[\phi_j + \phi_{jf}(t)] + x_i(t) \tag{3-14}$$

$$Q_1 = A_{jf}(t)\sin[\phi_j + \phi_{jf}(t)] + y_i(t) \tag{3-15}$$

$$P_2 = \pm A_s + A_{jf}(t - T_b)\cos[\phi_j + \phi_{jf}(t - T_b)] + x_{i-1}(t) \tag{3-16}$$

$$Q_2 = A_{jf}(t - T_b)\sin[\phi_j + \phi_{jf}(t - T_b)] + y_{i-1}(t) \tag{3-17}$$

In a manner like that used for analysis of DPSK systems in the Gaussian noise only case [1], the multiplier inputs are first written in the more convenient form:

$$d_i(t) = P_1\cos(w_c t) - Q_1\sin(w_c t) \tag{3-18}$$

$$d_{i-1}(t) = P_2\cos(w_c t) - Q_2\sin(w_c t) \tag{3-19}$$

The multiplier output is then

$$p(t) = d_i(t)d_{i-1}(t) \tag{3-20}$$

$$= P_1P_2\cos^2(w_c t) + Q_1Q_2\sin^2(w_c t) - [P_1Q_2 + P_2Q_1]\cos(w_c t)\sin(w_c t)$$

Thus

$$p(t) = \frac{1}{2}[P_1P_2 + Q_1Q_2] + \frac{1}{2}[P_1P_2 - Q_1Q_2]\cos(2w_c t)$$

$$- \frac{1}{2}[P_1Q_2 + P_2Q_1]\sin(2w_c t)$$

The lowpass filter eliminates the components about the frequency $2w_c$, yielding an output

$$\ell(t) = \frac{1}{2}[P_1P_2 + Q_1Q_2] \tag{3-21}$$

The receiver samples this output at time t_s and compares the result to the zero threshold. The decision rule of (2-5) and (2-6) can thus be written as, if at time t_s:

$$P_1P_2 + Q_1Q_2 \begin{array}{ll} > 0 & \text{decide } H_1 \\ < 0 & \text{decide } H_o \end{array} \tag{3-22}$$

An alternate form of the above rule is obtainable by use of the simple identity

$$ab + cd = \frac{1}{4}[(a + b)^2 + (c + d)^2 - (a - b)^2 - (c - d)^2] \tag{3-23}$$

which may be verified directly. Thus an equivalent form is: if at time t_s:

$$(P_1 + P_2)^2 + (Q_1 + Q_2)^2 \underset{H_o}{\overset{H_1}{\gtrless}} (P_1 - P_2)^2 + (Q_1 - Q_2)^2 \qquad (3\text{-}24)$$

where P_1, P_2, Q_1, and Q_2 for the case of Gaussian noise and FM interference are as given in (3-14) to (3-17). The complicated multiplications inherent in (3-22) have now been exchanged for squares of sum and difference quantities for which computations of probability density functions are more feasible, as will be shown subsequently.

Substituting the definitions (3-14) to (3-17), the decision rule is:

$$\{A_s + (\pm A_s) + A_{jf}(t_s)\cos[\phi_j + \phi_{jf}(t_s)] + A_{jf}(t_s - T_b) \qquad (3\text{-}25)$$

$$\times \cos[\phi_j + \phi_{jf}(t_s - T_b)] + x_i(t_s) + x_i(t_s - T_b)\}^2$$

$$+ \{A_{jf}(t_s)\sin[\phi_j + \phi_{jf}(t_s)] + A_{jf}(t_s - T_b)$$

$$\times \sin[\phi_j + \phi_{jf}(t_s - T_b)] + y_i(t_s) + y_i(t_s - T_b)\}^2$$

$$\underset{H_o}{\overset{H_1}{\gtrless}}$$

$$\{A_s - (\pm A_s) + A_{jf}(t_s)\cos[\phi_j + \phi_{jf}(t_s)] - A_{jf}(t_s - T_b)\cos[\phi_j + \phi_{jf}(t_s - T_b)]$$

$$+ x_i(t_s) - x_i(t_s - T_b)\}^2 + \{A_{jf}(t_s)\sin[\phi_j + \phi_{jf}(t_s)] - A_{jf}(t_s - T_b) \times$$

$$\times \sin[\phi_j + \phi_{jf}(t_s - T_b) + y_i(t_s) - y_i(t_s - T_b)\}^2$$

For notational simplicity, the following definitions are made:

$$A_1 = A_{jf}(t_s) \tag{3-26}$$

$$A_2 = A_{jf}(t_s - T_b) \tag{3-27}$$

$$\phi_1 = \phi_{jf}(t_s) \tag{3-28}$$

$$\phi_2 = \phi_{jf}(t_s - T_b) \tag{3-29}$$

$$\Delta\phi = \phi_2 - \phi_1 \tag{3-30}$$

In addition, it can now be noted that the decision rule of (3-25) contains random variables which consist of the sums and differences of independent zero mean Gaussian random variables with variance N, i.e., defining

$$U_1 = x_i(t_s) + x_i(t_s - T_b) \tag{3-31}$$

$$V_1 = y_i(t_s) + y_i(t_s - T_b) \tag{3-32}$$

$$U_2 = x_i(t_s) - x_i(t_s - T_b) \tag{3-33}$$

$$V_2 = y_i(t_s) - y_i(t_s - T_b) \tag{3-34}$$

then these random variables are also zero mean Gaussian random variables [22] with variance 2N.

Substituting the relations (3-26) to (3-34), the decision rule of (3-25) is:

$$\{A_s \pm A_s + A_1\cos(\phi_j + \phi_1) + A_2\cos(\phi_j + \phi_1 + \Delta\phi) + U_1\}^2 \tag{3-35}$$

$$+ \{A_1\sin(\phi_j + \phi_1) + A_2\sin(\phi_j + \phi_1 + \Delta\phi) + V_1\}^2$$

$$\underset{H_o}{\overset{H_1}{\gtrless}}$$

$$\{A_s \mp A_s + A_1\cos(\phi_j + \phi_1) - A_2\cos(\phi_j + \phi_1 + \Delta\phi) + U_2\}^2$$

$$+ \{A_1\sin(\phi_j + \phi_1) - A_2\sin(\phi_j + \phi_1 + \Delta\phi) + V_2\}^2$$

At this point, it can also be noted that since ϕ_j is uniformly distributed on the interval $(0,2\pi)$, then

$$\Phi = \phi_j + \phi_1 \tag{3-36}$$

$$= \phi_j + \phi_{jf}(t_s)$$

is also uniformly distributed modulo 2π over the same interval and may be substituted for same. Expanding again the cosine and sine arguments of (3-35) and making the further definitions

$$B = A_1 + A_2\cos(\Delta\phi) \tag{3-37}$$

$$C = A_2\sin(\Delta\phi) \tag{3-38}$$

$$D = A_1 - A_2\cos(\Delta\phi) \tag{3-39}$$

then the decision rule, in much simpler form, is:

$$\begin{aligned} &[A_s \pm A_s + B\cos\Phi - C\sin\Phi + U_1]^2 \\ &+ [B\sin\Phi + C\cos\Phi + V_1]^2 \\ &\mathop{\gtrless}_{H_o}^{H_1} \\ &[A_s \mp A_s + D\cos\Phi + C\sin\Phi + U_2]^2 \\ &+ [D\sin\Phi - C\cos\Phi + V_2]^2 \end{aligned} \tag{3-40}$$

The above representation of an equivalent decision rule illustrates the individual contributions of the signal, interference, and noise components to the decision process. For example, it can be seen via the definitions for B, C, and D, that the effect of the jamming interference is related to the values A_1 and A_2 of the jamming envelope at the sampling instants, and to the phase change $\Delta\phi$ between sampling instants.

Pr(err): Periodic FM Interference

The decision rule of (3-40) contains explicitly the five random variables Φ, U_1, V_1, U_2, and V_2. In addition, a sixth random variable is implicitly contained in the coefficients B, C, and D. This random variable is the offset time t_0 in the modulation m(t), which is assumed to be uniformly distributed in the interval $(0,T_m)$ and which enters into the determination of the envelope and phase parameters A_1, A_2, and $\Delta\phi$, via the Fourier Series coefficients a_n. The effect of this random variable will be suppressed for the present and attention centered on the effect of the five variables mentioned above.

By first defining the left hand side of (3-40) as R_1^2, the right hand side as R_0^2, and taking the square root of each side, an equivalent decision rule to (3-40), under hypothesis H_1, is:

$$R_1 = \{[2A_s + B\cos\Phi - C\sin\Phi + U_1]^2 + [B\sin\Phi + C\cos\Phi + V_1]^2\}^{1/2} \underset{H_o}{\overset{H_1}{\gtrless}} \tag{3-41}$$

$$R_o = \{[D\cos\Phi + C\sin\Phi + U_2]^2$$

$$+ [D\sin\Phi - C\cos\Phi + V_2]^2\}^{1/2}$$

Under the hypothesis H_1 and the above decision rule, an error must occur if $R_1 < R_o$, thus

$$\Pr(err | H_1) = \Pr(R_1 < R_o) \tag{3-42}$$

The task remaining is an evaluation of (3-42) for the R_1 and R_o of (3-41). Several results from the literature are used in the following for brevity, whereas the complete derivation is available in the Appendix for reference. First defining the random variables:

$$\alpha_1 = D\cos\Phi + C\sin\Phi + U_2 \tag{3-43}$$

$$\alpha_2 = D\sin\Phi - C\cos\Phi + V_2 \tag{3-44}$$

then conditional on ϕ_j, $\phi_{jf}(t_s)$, C, and D; the random variables α_1 and α_2 are independent Gaussian variables with means

$$n_1 = D\cos\Phi + C\sin\Phi \tag{3-45}$$

$$n_2 = D\sin\Phi - C\cos\Phi \tag{3-46}$$

and identical variances

$$\sigma^2 = \sigma_1^2 = \sigma_2^2 = 2N \tag{3-47}$$

The random variable R_o is thus seen to be the square root of the sum of squares of two independent Gaussian variables with the indicated means and variances. Thus R_o is a Rician variable [22] with probability density function:

$$f(r_o|H_1) = (r_o/\sigma^2)\exp[-(r_o^2 + n_1^2 + n_2^2)/2\sigma^2]I_o[r_o\sqrt{n_1^2 + n_2^2}/\sigma^2] \tag{3-48}$$

where $I_o(X)$ is the modified Bessel function of the first kind and zero order [25], i.e.

$$I_o[(a^2 + b^2)^{1/2}] = (1/2\pi)\int_o^{2\pi}\exp[a\cos z + b\sin z]dz \tag{3-49}$$

and

$$n_1^2 + n_2^2 = C^2 + D^2 \tag{3-50}$$

$$\sigma^2 = 2N \tag{3-51}$$

The density function for R_1 can be determined in a similar manner since under the hypothesis H_1 we have by defining

$$\beta_1 = 2A_s + B\cos\Phi - C\sin\Phi + U_1 \qquad (3\text{-}52)$$

$$\beta_2 = B\sin\Phi + C\cos\Phi + V_1 \qquad (3\text{-}53)$$

that, conditional on ϕ_j, $\phi_{jf}(t_s)$, B, and C; the variables β_1 and β_2 are also independent Gaussian variables with means

$$n_1 = 2A_s + B\cos\Phi - C\sin\Phi \qquad (3\text{-}54)$$

$$n_2 = B\sin\Phi + C\cos\Phi \qquad (3\text{-}55)$$

and identical variances:

$$\sigma^2 = \sigma_1^2 = \sigma_2^2 = 2N \qquad (3\text{-}56)$$

Via the previous discussion for R_o, it follows that R_1 is also Rician with density function

$$f(r_1|H_1) = (r_1/\sigma^2)\exp[-(r_1^2 + n_1^2 + n_2^2)/2\sigma^2]I_o[r_1\sqrt{n_1^2 + n_2^2}/\sigma^2] \qquad (3\text{-}57)$$

in which

$$n_1^2 + n_2^2 = 4A_s^2 + 4A_s(B\cos\Phi - C\sin\Phi) + C^2 + B^2 \qquad (3\text{-}58)$$

$$\sigma^2 = 2N \qquad (3\text{-}59)$$

The error equation (3-42) is thus the probability that one Rician variable is greater than another, each of whose density functions are of the form:

$$f(r_i|H_1) = (r_i/\sigma_i^2)\exp[-(r_i^2 + a_i^2)/2\sigma_i^2]I_o[r_i a_i/\sigma_i^2] \tag{3-60}$$

A result from the literature [26][27] is:

$$\Pr(R_1<R_o) = Q[\sqrt{a},\sqrt{b}] - (c^2/(1 + c^2))\exp[-(a + b)/2]I_o[\sqrt{ab}] \tag{3-61}$$

$$= \Pr(\text{err}|H_1,\Phi)$$

where

$$a = a_o^2/(\sigma_o^2 + \sigma_1^2) \tag{3-62}$$

$$b = a_1^2/(\sigma_o^2 + \sigma_1^2) \tag{3-63}$$

$$c = \sigma_1/\sigma_2 \tag{3-64}$$

$$Q[\alpha,\rho] = \int_\rho^\infty t\exp[-(t^2 + \alpha^2)/2]I_o(\alpha t)dt \tag{3-65}$$

is the Marcum Q-function [26], and the conditional dependence on Φ has been explicitly indicated. Thus $\Pr(\text{err}|H_1,\Phi)$ is given by (3-61) in which

$$a = (C^2 + D^2)/4N \tag{3-66}$$

$$b = (4A_s^2 + 4A_s(B\cos\Phi - C\sin\Phi) + C^2 + B^2)/4N \tag{3-67}$$

$$c = 1 \tag{3-68}$$

Under hypothesis H_o, the decision rule of (3-40) becomes

$$R_1 = \{[B\cos\Phi - C\sin\Phi + U_1]^2 + [B\sin\Phi + C\cos\Phi + V_1]^2\}^{1/2} \tag{3-69}$$

$$\underset{H_o}{\overset{H_1}{\gtrless}}$$

$$R_o = \{[2A_s + D\cos\Phi + C\sin\Phi + U_2]^2 + [D\sin\Phi - C\cos\Phi + V_2]^2\}^{1/2}$$

and

$$\Pr(err|H_o) = \Pr(R_o < R_1) \tag{3-70}$$

By direct comparison of (3-69) and (3-70) with the corresponding (3-41) and (3-42), it can be observed that the error calculation for $\Pr(err|H_o)$ is identical to that just completed for $\Pr(err|H_1)$, if only D is replaced by B, C by (-C), and B by D in (3-41). Thus making these replacements in (3-66) and (3-67), we immediately have from (3-61) that

$$Pr(R_o < R_1) = Pr(err|H_o,\Phi) \tag{3-71}$$

$$= Q[\sqrt{a},\sqrt{b}] - (c^2/(1 + c^2))\exp[-(a + b)/2]I_o[\sqrt{ab}]$$

where now

$$a = (C^2 + B^2)/4N \tag{3-72}$$

$$b = (4A_s^2 + 4A_s(D\cos\Phi + C\sin\Phi) + C^2 + D^2)/4N \tag{3-73}$$

$$c = 1 \tag{3-74}$$

and the conditional dependence on Φ has again been explicitly indicated. Since Φ is a uniform random variable via (3-36) we have:

$$Pr(err|H_1) = (1/2\pi)\int_o^{2\pi} Pr(err|H_1,\Phi)d\Phi \tag{3-75}$$

$$Pr(err|H_o) = (1/2\pi)\int_o^{2\pi} Pr(err|H_o,\Phi)d\Phi \tag{3-76}$$

Substituting the relations (3-61), (3-66) to (3-68), (3-71), (3-72) to (3-74), (3-37) to (3-39):

$$\Pr(\text{err}|H_1,x) = \frac{1}{2\pi}\int_o^{2\pi}\Bigg\{ Q\Bigg[\left[\frac{A_1^2 - 2A_1A_2\cos\Delta\phi + A_2^2}{4N}\right]^{1/2}, \tag{3-77}$$

$$\left[\frac{4A_s^2 + A_1^2 + 2A_1A_2\cos\Delta\phi + A_2^2 + 4A_s(A_1\cos y_3 + A_2\cos(y_3 + \Delta\phi)}{4N}\right]^{1/2}\Bigg]$$

$$- \frac{1}{2}\exp[-(4A_s^2 + 2(A_1^2 + A_2^2) + 4A_s(A_1\cos y_3 + A_2\cos(y_3 + \Delta\phi)))/8N]$$

$$\times I_o[(4A_s^2 + A_1^2 + 2A_1A_2\cos\Delta\phi + A_2^2 + 4A_s(A_1\cos y_3$$

$$+ A_2\cos(y_3 + \Delta\phi))^{1/2}(A_1^2 - 2A_1A_2\cos\Delta\phi + A_2^2)^{1/2}/4N]\Bigg\}\, dy_3$$

$$\Pr(\text{err}|H_o,x) = \frac{1}{2\pi}\int_o^{2\pi}\Bigg\{ Q\Bigg[\left[\frac{A_1^2 + 2A_1A_2\cos\Delta\phi + A_2^2}{4N}\right]^{1/2}, \tag{3-78}$$

$$\left[\frac{4A_s^2 + A_1^2 - 2A_1A_2\cos\Delta\phi + A_2^2 + 4A_s(A_1\cos y_3 - A_2\cos(y_3 + \Delta\phi)}{4N}\right]^{1/2}\Bigg]$$

$$- \frac{1}{2}\exp[-(4A_s^2 + 2(A_1^2 + A_2^2) + 4A_s(A_1\cos y_3 - A_2\cos(y_3 + \Delta\phi)))/8N]$$

$$\times I_o[(4A_s^2 + A_1^2 - 2A_1A_2\cos\Delta\phi + A_2^2 + 4A_s(A_1\cos y_3$$

$$- A_2\cos(y_3 + \Delta\phi))^{1/2}(A_1^2 + 2A_1A_2\cos\Delta\phi + A_2^2)^{1/2}/4N]\Bigg\}\, dy_3$$

where from equations (3-26) to (3-30), with the sampling time t_s set arbitrarily at T_b:

$$A_1 = A_{jf}(T_b, x) \tag{3-79}$$

$$A_2 = A_{jf}(0, x) \tag{3-80}$$

$$\phi_1 = \phi_{jf}(T_b, x) \tag{3-81}$$

$$\phi_2 = \phi_{jf}(0, x) \tag{3-82}$$

$$\Delta\phi = \phi_2 - \phi_1 \tag{3-83}$$

$$x = 2\pi t_o/T_m \tag{3-84}$$

In the above, the previous suppression of the offset parameter t_0 has been removed, since the amplitudes, A_1 and A_2, and phases, ϕ_1 and ϕ_2, of the filtered interference at the sampling instants depend on the value of this parameter, which is uniformly distributed on $(0,T_m)$. Through the definition (3-84), the dependence of the error probabilities on t_0 is related to an equivalent random variable x, which via (3-84) is uniformly distributed on the interval $(0,2\pi)$.

The total error probability, conditioned on x, is

$$\Pr(\text{err}|x) = P_1 \Pr(\text{err}|H_1,x) + P_o \Pr(\text{err}|H_o,x) \tag{3-85}$$

where P_1 and P_0 are the respective a priori probabilities of H_1 and H_0 being true. The dependence on the random variable x is eliminated by averaging the error probability over the values of x, i.e.,

$$\Pr(\text{err}) = \int_0^{2\pi} \Pr(\text{err}|x) f(x)dx \tag{3-86}$$

where the density function of x is

$$f(x) = 1/2\pi \qquad 0 \leq x \leq 2\pi \tag{3-87}$$

Substitution of (3-85) and (3-87) into (3-86) yields

$$\Pr(\text{err}) = (1/2\pi) \int_0^{2\pi} [P_1 \Pr(\text{err}|H_1,x) + P_o \Pr(\text{err}|H_o,x)]dx \tag{3-88}$$

into which, the very complex relations (3-77) and (3-78) must be substituted for evaluation of the error probability.

Even though equation (3-88) is already quite involved, one item of consideration yet remains. This is the dependence of the error probability on the sampling times of zero and T_b seconds. The parameters A_1, A_2, and $\Delta\phi$ are contained in the integrand of (3-88). If these parameters are not identical in each and every signaling interval, then equation (3-88) represents an error probability which of necessity changes from one bit to the next, and thus does not represent an average error rate for the system. In the most general case, the filtered envelope and phase of the interference cannot be expected to return at time T_b to their initial values at the start of the interval, and thus one evaluation of (3-88) is not sufficient.

In this case, the individual bit error probabilities of (3-88) must be calculated over an infinite sequence of signaling intervals and averaged for a final average error probability. Thus re-defining equations (3-79) to (3-84) as

$$A_1 = A_{jf}(nT_b, x) \tag{3-89}$$

$$A_2 = A_{jf}((n-1)T_b, x) \tag{3-90}$$

$$\phi_1 = \phi_{jf}(nT_b, x) \tag{3-91}$$

$$\phi_2 = \phi_{jf}((n-1)T_b, x) \tag{3-92}$$

$$\Delta\phi = \phi_2 - \phi_1 \tag{3-93}$$

then the system error probability is

$$\Pr(\text{err}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (1/2\pi) \int_o^{2\pi} [P_1 \Pr(\text{err}|H_1, x) + P_o \Pr(\text{err}|H_o, x)]dx \tag{3-94}$$

where $\Pr(\text{err}|H_1, x)$ and $\Pr(\text{err}|H_0, x)$ are as given in (3-77) and (3-78) with the redefinitions (3-89) to (3-93).

Equation (3-94), while mathematically correct for the general case, would require massive numerical computation to obtain quantitative results. For this reason, the analysis will henceforth consider a particular class of FM interferences for which the computational difficulties are reduced, but which can be used to approximate the general case as closely as desired.

Pr(err): FM Interference with Rational Slip Ratio and Frequency Offset

The question arises as to whether the error form of (3-94) is always required, or whether there is a simplification possible. This question is fundamentally related to the question of whether the values of the envelope and phase of the interference return to their initial values at the end of some integral number, p, of signaling intervals. If so, the infinite summation indicated in (3-94) is replaceable by a finite summation over N = p signaling intervals. The question then is: What restrictions must be placed on the form of the interference so that the envelope and phase are periodic in an integral number of signaling intervals, and how severe are these restrictions?

Returning to equation (3-5) and (3-6) for the time-varying envelope and phase function, the dependence on time lies solely in the parameter $\rho_n(t)$ as defined in (3-1). Thus if $\rho_n(t)$ is periodic then the envelope and phase are certainly periodic with the same period.

A determination of the period of $\rho_n(t)$ requires an analysis of its frequency content. Immediately from (3-1), the set of frequency components $\{w_{e,\phi}\}$ contained in $\rho_n(t)$ is

$$\{w_{e,\phi}\} = w_j - w_c + nw_m \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3-95}$$

Defining the frequency offset parameter w_{fo} as

$$w_{fo} = (w_j - w_c)/w_m \tag{3-96}$$

we now consider the class of FM interference for which the frequency offset is a rational number, i.e.,

$$w_{fo} = N + (u/v) \qquad N = 0, \pm 1, \pm 2, \ldots \tag{3-97}$$

where u and v are nonnegative integers, with u < v.

In this case, the set of frequencies in $\rho_n(t)$ is

$$\{w_{e,\phi}\} = \{[N + n + (u/v)]w_m\} \tag{3-98}$$

Expanding by substitution of the possible values of N and n, the frequency set is:

$$\{w_{e,\phi}\} = \{[u/v,\ 1 + (u/v),\ -1 + (u/v),\ \ldots,\ \pm K + (u/v)]w_m\} \tag{3-99}$$

Since $u \leq v$, the lowest frequency component in $\rho_n(t)$ is required to be one of the two cases:

$$(w_{e,\phi})_{min} = (u/v)w_m \qquad u/v \leq 1/2 \tag{3-100}$$

$$(w_{e,\phi})_{min} = [(-1 + (u/v)]w_m \qquad u/v > 1/2 \tag{3-101}$$

Although one of these two frequencies is the lowest contained in $\rho_n(t)$, the period is not directly related to this lowest frequency by virtue of the fact that the other frequency components are not necessarily harmonics of the lower frequency, e.g., consider an offset of $.4w_m$, for which:

$$N = 0$$

$$u/v = 2/5$$

$$\{w_{e,\phi}\} = \{[2/5,\ 7/5,\ -3/5,\ 12/5,\ -8/5,\ \ldots]w_m\}$$

$$(w_{e,\phi})_{min} = (2/5)w_m$$

This set contains other than integral multiples of the minimum frequency, so that the period $T_{e,\phi}$ corresponding to $(w_{e,\phi})_{min}$ cannot be considered as the fundamental period of the envelope and phase functions. It is true, however, that the frequency

$$w_{e,\phi} = w_m/v \tag{3-102}$$

can be considered as the fundamental frequency, so that by choosing

$$T_{e,\phi} = vT_m \tag{3-103}$$

then the envelope and phase will be periodic with period $T_{e,\phi}$, i.e.,

$$A_{jf}(t) = A_{jf}(t + vT_m) \tag{3-104}$$

$$\phi_{jf}(t) = \phi_{jf}(t + vT_m) \tag{3-105}$$

Thus in the preceding example, the fundamental frequency can be taken as $(w_m/5)$, thus insuring repetition of the envelope and phase every $5T_m$ seconds.

In conclusion then, it is obvious that: <u>for any rational frequency offset, the filtered envelope and phase must be repetitive at some integral multiple of the modulation period T_m.</u>

For the envelope and phase to repeat q times every p signaling intervals, then it is required that

$$qT_{e,\phi} = pT_b \qquad q = 1, 2, 3, \ldots \tag{3-106}$$

using the relation (3-103), we have that

$$qvT_m = pT_b \tag{3-107}$$

or

$$T_m/T_b = p/(qv) \tag{3-108}$$

Defining the ratio of the signaling period to the modulation period as the <u>slip ratio</u>

$$S_r = T_b/T_m = v/(p/q) \tag{3-109}$$

it is obvious that the slip ratio must be a rational number if (3-106) is to hold. Thus given a rational slip ratio, the relation

$$p/q = v/S_r = v(T_m/T_b) \tag{3-110}$$

must hold, i.e., if the slip ratio S_r and the frequency offset are known rational numbers, then in p signaling intervals exactly q periods of the envelope and phase will occur, in accordance with (3-110). For example, if:

$$T_m/T_b = 5/6$$

$$w_{fo} = 1/2$$

then

$$v = 2$$

$$p/q = 2(5/6) = 5/3$$

and the envelope and phase will pass through exactly three cycles every five bit intervals, repeating indefinitely. Thus (3-94) can be truncated to an averaging over only the five bit intervals.

The requirement that the slip ratio and frequency offset be rational numbers is not excessively restrictive, since any irrational number can be approximated to any arbitrary tolerance by choice of the proper rational number.

Pr(err): Integer Values of Slip Ratio and Frequency Offset

The class of interference for which

$$w_{fo} = (w_j - w_c)/w_m = K \qquad K \text{ an integer} \tag{3-111}$$

$$S_r = j \qquad j \text{ a positive integer} \tag{3-112}$$

will now be considered. Since from the definition (3-109) of S_r we have an alternative form:

$$S_r = w_m/w_b \tag{3-113}$$

then the product of slip ratio and frequency offset is:

$$w_{fo}S_r = (w_j - w_c/w_m)(w_m/w_b) \tag{3-114}$$

i.e.,

$$(w_j - w_c)/w_b = Kj \tag{3-115}$$

Comparing (3-114) and (3-115) it is obvious that this class of interference includes the important subset of cases for which the jamming carrier is cochannel with the signal carrier ($K = 0$), band-edge ($Kj = 1$), and interchannel ($Kj > 1$).

Comparison of (3-111) with (3-97) and (3-99) yields the result that the fundamental frequency of the envelope and phase functions is w_m, and thus that

$$u/v = 0$$

$$v = 1 \qquad \text{(arbitrary)}$$

$$T_{e,\phi} = T_m \tag{3-116}$$

for this class of interference. From (3-110) we then have that

$$p/q = 1/S_r = 1/j \tag{3-117}$$

Thus in accordance with the previous discussion, the envelope and phase will repeat themselves j times <u>each</u> bit interval, and the initial and final values each bit interval are identical.

As a result, the error equation (3-94) simplifies to the result of (3-88) and now:

$$A_1 = A_2 = A_{jf}(0,x) \tag{3-118}$$

$$\Delta\phi = \phi_1 - \phi_2 = 0 \tag{3-119}$$

Returning to the error probabilities of equations (3-77) and (3-78), considerable simplification is now obtainable. Substituting (3-118) and (3-119), we have:

$$\Pr(\mathrm{err}|H_1,x) = (1/2\pi)\int_o^{2\pi}\{Q[0,((4A_s^2 + 4A_1^2 + 8A_sA_1\cos y_3)/4N)^{1/2}] \tag{3-120}$$

$$- \frac{1}{2}\exp\,[-(4A_s^2 + 4A_1^2 + 8A_sA_1\cos y_3)/8N]I_o[0]\}dy_3$$

$$\Pr(\mathrm{err}|H_o,x) = (1/2\pi)\int_o^{2\pi}\{Q[A_1/\sqrt{N},\ A_s/\sqrt{N}] - \frac{1}{2}\exp\,[-(A_s^2 + A_1^2)/2N] \tag{3-121}$$

$$\times\ I_o[A_sA_1/N]\}dy_3$$

Using the definition (3-49) of $I_0(\cdot)$, and the relation [26] that

$$Q[0,\rho] = \exp[-\rho^2/2]$$

then we have from (3-120) that

$$\Pr(\mathrm{err}|H_1,x) = (1/4\pi)\int_o^{2\pi}\exp[-(4A_s^2 + 4A_1^2 + 8A_sA_1\cos y_3)/8N]dy_3 \tag{3-122}$$

$$= \frac{1}{2}\exp\,[(-4A_s^2 + 4A_1^2/8N]\{(1/2\pi)\int_o^{2\pi}\exp[-A_sA_1\cos y_3/N]dy_3\}$$

i.e.,

$$\Pr(\mathrm{err}|H_1,x) = \frac{1}{2}\exp\,[-(A_s^2 + A_1^2)/2N]I_o[A_sA_1/N] \qquad (3\text{-}123)$$

which is the final result for the error probability conditioned on H_1.

Inspection of (3-121) reveals that the variable of integration y_3 does not now appear in the integrand, thus immediately integrating, we have:

$$\Pr(\mathrm{err}|H_o,x) = Q[A_1/\sqrt{N},\ A_s/\sqrt{N}] - \frac{1}{2}\exp\,[-(A_s^2 + A_1^2/2N)] \qquad (3\text{-}124)$$

$$\times\, I_o[A_sA_1/N]$$

which is the final result for the error probability conditioned on H_o.

Thus upon substitution of (3-123) and (3-124) into (3-88), the total system error is

$$\Pr(\mathrm{err}) = (1/2\pi)\int_o^{2\pi}\{P_oQ[A_1/\sqrt{N},\ A_s/\sqrt{N}] + \frac{1}{2}(P_1 - P_o) \qquad (3\text{-}125)$$

$$\times\exp[-(A_s^2 + A_1^2)/2N]I_o[A_sA_1/N]\}dx$$

Under the usual assumption of equally likely transmission of ones and zeros, the system error reduces to:

$$\Pr(\mathrm{err}) = (1/4\pi)\int_o^{2\pi}Q[A_1/\sqrt{N},\ A_s/\sqrt{N}]dx \qquad (3\text{-}126)$$

since $P_1 = P_0 = 1/2$.

As a result of the complex dependence of A_1 on the random variable x through the Fourier Series variables a_n and ϕ_n, as related by equations (2-20), (2-23), (3-1), (3-5), (3-84), and (3-118), the integration in the final result of (3-126) to a closed form is not feasible. Quantitative results must thus be obtained via numerical integration techniques. Important results for a number of interference types of the rational group are discussed in the following chapter.

Pr(err): Zero Sweep (CW) Interference

When the sweep modulation m(t) is zero, the FM interference of (2-16) simplifies to a sinusoid of amplitude A_j, frequency w_j, and random phase ϕ_j, i.e.,

$$[j(t)]_{cw} = A_j \cos(w_j t + \phi_j) \tag{3-127}$$

so that the filtered version is

$$[j_f(t)]_{cw} = A_j A(w_j) \cos(w_j t + \phi_j + \theta(w_j)) \tag{3-128}$$

and the envelope and phase functions are constants for all time. Thus

$$\Pr(err)_{cw} = (1/4\pi) \int_o^{2\pi} Q[A_{jf}/\sqrt{N},\ A_s/\sqrt{N}]dx \tag{3-129}$$

where

$$A_{jf} = A_j A(w_j) \tag{3-130}$$

is independent of x. Thus for CW interference, the general results previously obtained reduce to the form

$$\Pr(\text{err})_{cw} = \frac{1}{2} Q \; [A_j A(w_j)/\sqrt{N},\; A_s/\sqrt{N}] \tag{3-131}$$

after integrating in (3-129) over the variable x. This result, a degenerate solution of the general FM case, has been previously reported [13]. The error rate for CW as given by (3-131), is however, useful for comparison with the types of FM interference considered in the next chapter.

CHAPTER IV

QUANTITATIVE METHODS, RESULTS, AND CONCLUSIONS

As noted in the preceding chapter the involved integrations required for obtaining quantitative data from the error equations of the preceding chapter prevent solutions of a closed form. As a result, numerical methods are required to generate quantitative data useful for comparing DPSK systems with different bandpass filters and/or different interfering types.

Computational Method

In order to accommodate the extensive number of parameters which must be used to adequately define the signal, interference, power ratios, and bandpass filters, a computer program consisting of a compact main program and numerous subroutines has been implemented. The fundamental flow of operation is the following:

(a) A choice of linear, sinusoidal, or CW frequency modulated interference is made in the main program, and the parameter values for modulation index, slip ratio, and frequency offset specified. A switch variable is used to call up the subroutine associated with the interference type, and the Fourier parameters $|a_n|$ and ϕ_n of equation (2-25) are calculated. The equations defining these parameters are first analytically derived (see APPENDIX I for linear FM) using equations (2-20), (2-22) and (2-23). The infinite array of coefficients is truncated when the power contained in the series expansion of equation (2-25) exceeds 99% of the defined power level. In addition, a calculation of the envelope of the series expansion is used to check for envelope constancy of the unfiltered interference. Alternately, the envelope calculation can be called upon for observation of envelope variation (amplitude modulation) generated by the bandpass filtering process.

(b) The choice of bandpass filter type to be employed is made in the main program and an associated subroutine similarly called. Specified inputs to the subroutine consist of the parameter values defining filter order, half power bandwidth, and interfering frequency offset and slip ratio. The generated output consists of an array of attenuation and phase coefficients, $A(w_j + nw_m)$ and $\theta(w_j + nw_m)$ as required in equation (2-26). The following types of bandpass filters are initially available for call by the main program: (1) Nth order Butterworth; (2) First and fourth order Chebyshev with 0.5 db ripple; (3) Ideal with $A(f_{co}) = .707$. For unified comparison, the half-power bandwidths of these filter types are defined to be twice the signaling frequency ($2/T_b$). In addition another ideal filter subroutine with $A(f_{co}) = 1$, and $A(f) = 0$ for $f > f_{co}$, is available.

(c) Initial values of signal-to-noise and signal-to-interference power ratios are specified and a numerical integration subroutine is called upon for evaluation of the desired error equation. Associated with this call-up are other subroutine calls which define and calculate the integrand of interest. Among these are a modified version of a Q-function subroutine developed by Johansen [28], and filtered envelope and phase subroutines required for evaluation of equations (3-5) and (3-6). The Q-function calculation has a relative accuracy of one part in 10^5 and the integration subroutine a relative accuracy of one part in 10^3.

(d) Looping in the main program is incorporated for iterative calculations of system error when specified parameters are value incremented.

Cochannel Interference

A considerable portion of the investigation has been concerned with determining DPSK system susceptibility to jamming types in which the interfering carrier and signal carrier are cochannel, i.e., identical frequencies. In addition, direct comparisons of the jamming capability of the CW, linear, and sinusoidal FM interferences were desired for this case, as were conclusions concerning the effects of the different bandpass filters previously described.

In order to establish some reference levels for which comparative analyses can be made and conclusions obtained, it is first necessary to define operating environments for the system which will serve this purpose. The primary reference point has been chosen to be the Gaussian-noise-only environment, for which the signal-to-interference power ratio is infinite. Furthermore, as derived in the previous chapter, CW interference is a special, degenerate case of the more general frequency modulated class of interferers, and the system error for cochannel CW interference is well known [13]. Thus the system error arising from CW interference has been chosen as a secondary reference in describing system operation in the presence of linear and sinusoidal FM interferences. For cross-comparisons between the numerous interference combinations available via parameter variation and filter type, a tertiary operating reference has been defined. This reference level is chosen as that system error resulting when the DPSK system is subjected to FM interference with unity modulation index and slip ratio, and when the bandpass filter is a first-order Butterworth of the type previously mentioned.

<u>Pr(err): First-order Butterworth Filtering</u>

<u>Linear FM</u>. Figure 3 illustrates the dependence of system error on signal to noise ratio (SNR) with signal-to-interference ratio (SIR) as a parameter, and with unity modulation index (β) and slip

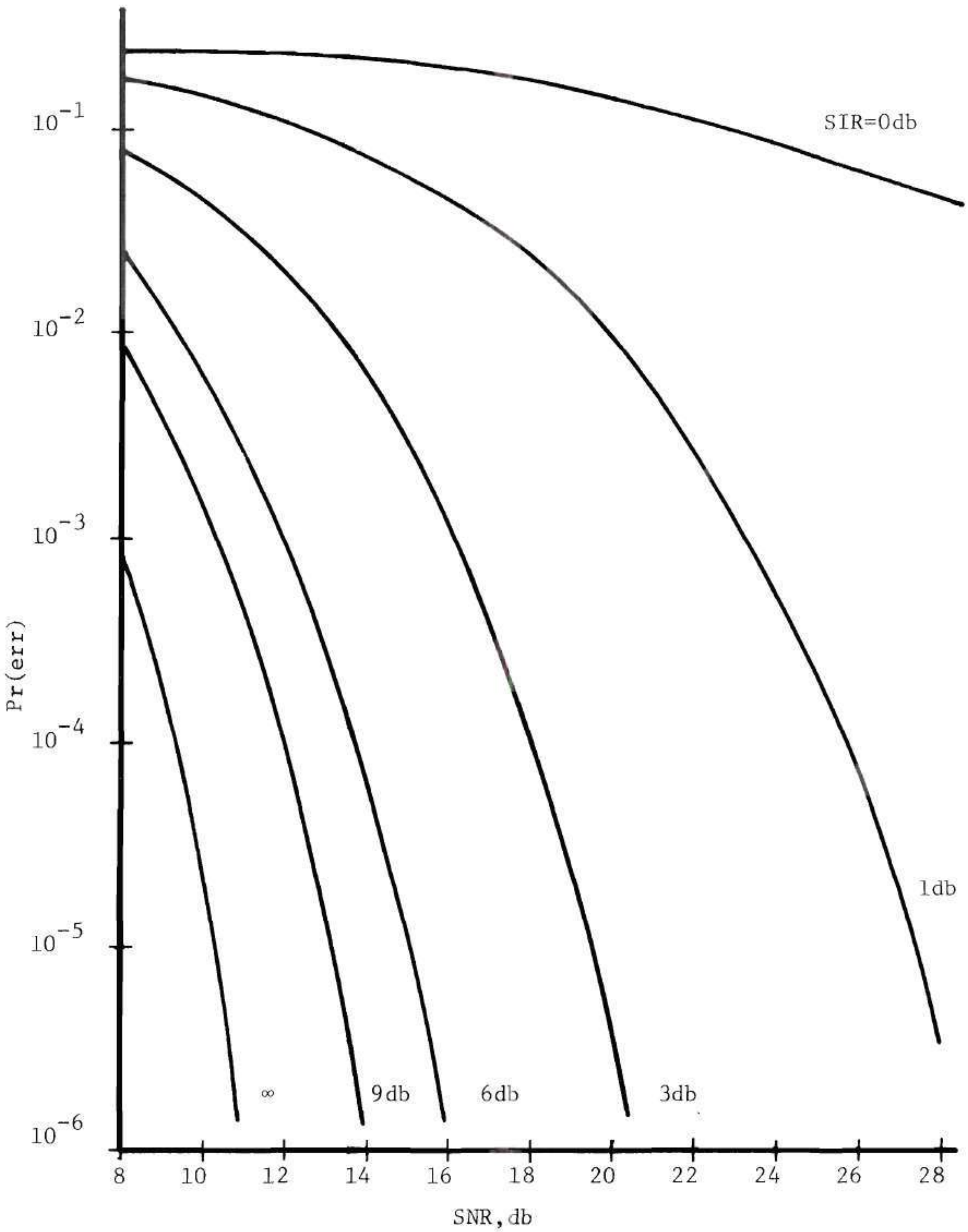

Figure 3. Cochannel Error Rate vs SNR and SIR: Linear FM, β=1

ratio (S_r). The rapid increase in system error, from the Gaussian-noise-only case (SIR = ∞) to the "jammed" case for which the SIR is zero db, is quite evident. For example, an unjammed DPSK system operating satisfactorily with an SNR of 10db and error rate of 2×10^{-5} will deteriorate rapidly to an error rate of only 5×10^{-2} for an SIR of 3db. This family of curves can be used to determine the increase in signal power required for system recovery in the presence of interference. For the case just mentioned a 4db increase in signal power will return the error rate to approximately 2×10^{-5} (SNR = 14db, SIR = 7db).

Sinusoidal FM. Figure 4 displays similar error curves obtained for sinusoidal modulation, with all other parameters identical to that of Figure 3. The effect of increasing jammer power is again obvious. For the unjammed system previously cited (SNR = 10db, Pr(err) = 2×10^{-5}), the system error rate increases to approximately 2×10^{-2} for an SIR 3db. As in the case of linear FM, a 4db increase in signal power will return the system near its initial error rate. By direct comparison of Figure 4 with Figure 3 at identical SNR and SIR, it can be observed that linear FM is somewhat superior to sinusoidal FM, the relative superiority increasing with increase in jamming level (decrease in SIR). This property is discussed in more detail in subsequent sections.

Spot versus Barrage Jamming. As previously noted in CHAPTER I, a useful property of FM interference is that a simple variation of the modulation index β allows the interfering source to easily range

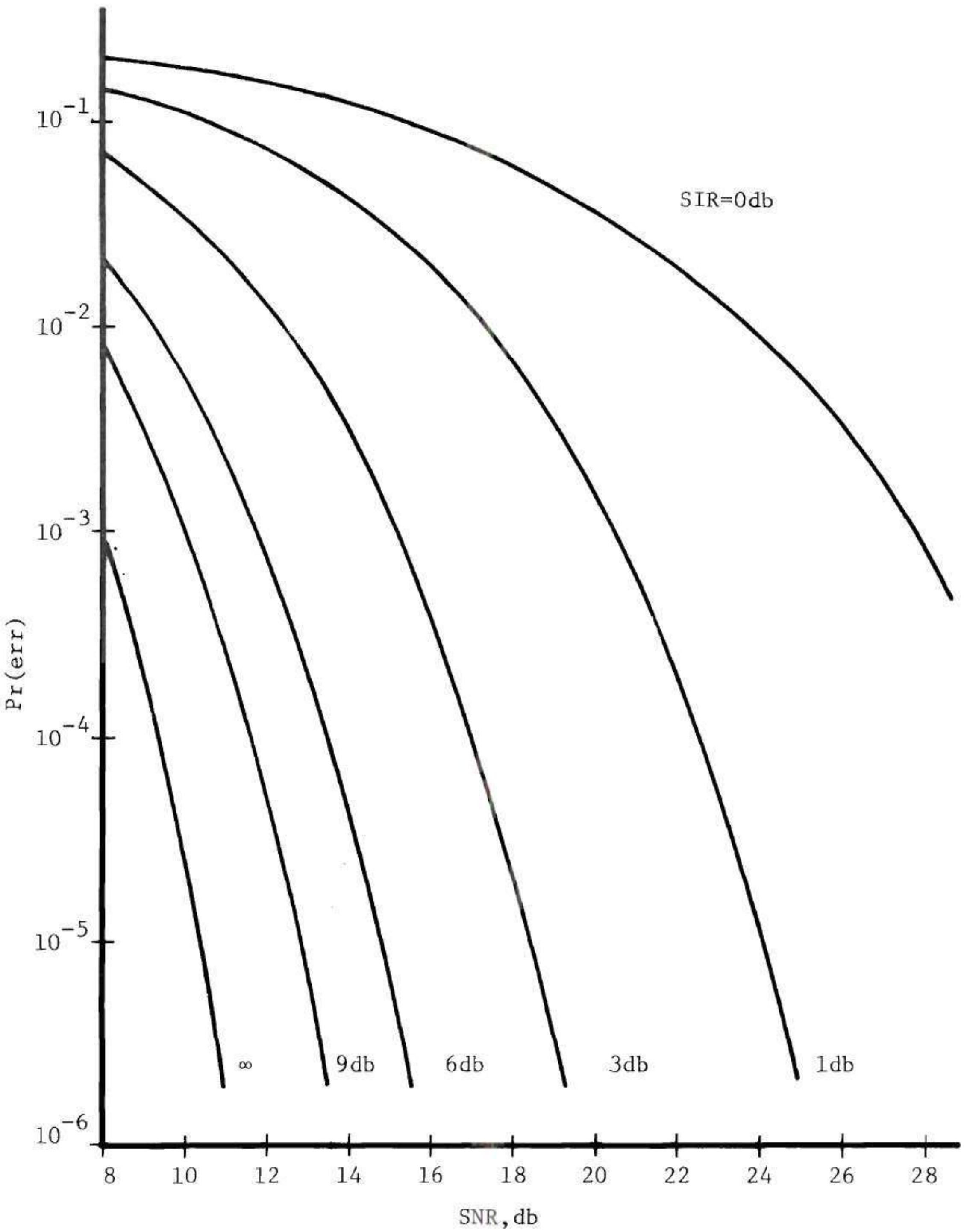

Figure 4. Cochannel Error Rate vs SNR and SIR: Sinusoidal FM, β=1

from a CW spot jamming mode ($\beta = 0$) to a barrage jamming mode ($\beta > 1$). To investigate the corresponding effects on system susceptibility, numerous determinations of error rate with varying modulation indices have been made.

Figure 5 displays the system error rate versus SNR obtained for linear FM, with modulation index as a parameter. The SIR for this family of curves is 3db. As indicated in the figure, the error rate has been found to decrease monotonically with increasing modulation index. The CW cochannel interference can be observed to cause maximum error rate. Thus for the first-order Butterworth (BW1) filter, spreading of the power spectrum by a cochannel interfering source results in a loss of jamming effectiveness, assuming constant SIR (no increase in jammer power).

The effect of different SIR levels upon curves of this type has also been determined. In comparing the results displayed in Figure 5 with similar families of curves with larger SIR, it was observed that the spreading of the error family (from $\beta = 0$ to $\beta = 4$) decreases with increasing SIR. Considering Figure 5 for example, and using the unity modulation index case as the reference norm, a decrease in β to zero requires an increase in SNR of approximately 1db if the $\beta = 1$ error rate is to be maintained, i.e., the curve is shifted 1db to the right. An increase to $\beta = 4$ from $\beta = 1$ corresponds to a decrease in required SNR of approximately 3db, i.e., a 3db shift of the curve to the left. For a larger SIR level of 9db the corresponding required figures are an increase in SNR of only 0.2db and a decrease in SNR of

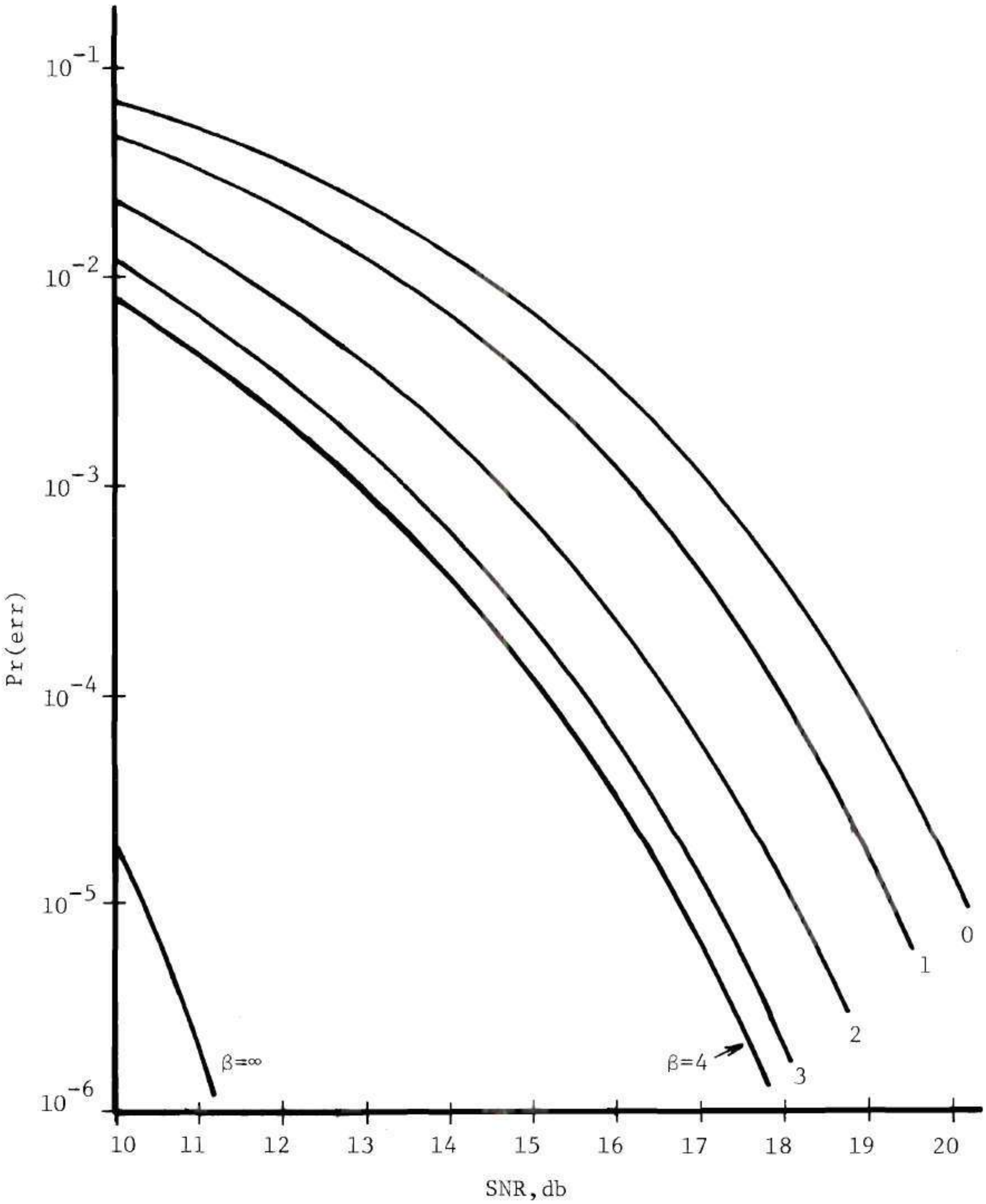

Figure 5. Cochannel Error Rate vs SNR and Modulation Index: Linear FM, SIR=3db

only 1db. The spread of the curves with β is thus measurably reduced in the latter case of higher SIR. Furthermore, by determining similar values for different values of β and SIR it is possible to use Figure 3 as a universal error rate family for linear FM interference.

Figure 6 is a plot of required SNR increase versus modulation index with SIR as a parameter. The SNR increase is relative to the reference curves for unity modulation index of Figure 3, thus all the SIR curves of Figure 6 pass through zero for unity modulation index. Figure 6 thus serves to specify the SNR shift of the reference curves of Figure 3 when the modulation index is varied, at a specified SIR. For example, the error curve for an SIR of 6db with $\beta = 3$ can be obtained by first noting from Figure 6 that a decrease in SNR of approximately 1.3db is allowable for the same error rate as that obtained for $\beta = 1$, thus the desired curve lies 1.3db to the left of the SIR = 6db curve of Figure 3.

A similar technique can be used for specifying the system error rates for sinusoidal FM with non-unity modulation index. Figure 7 is a corresponding family of SNR shift curves which can be used in conjunction with the sinusoidal FM reference curves of Figure 4 to specify error rates for non-unity values of modulation index.

As mentioned previously, cochannel linear FM interference is more effective as a jamming source than cochannel sinusoidal FM interference. This conclusion may be validated by use of the preceding graphical techniques for any similar values of β and SIR, as has been verified by hundreds of exact Pr(err) calculations and comparisons.

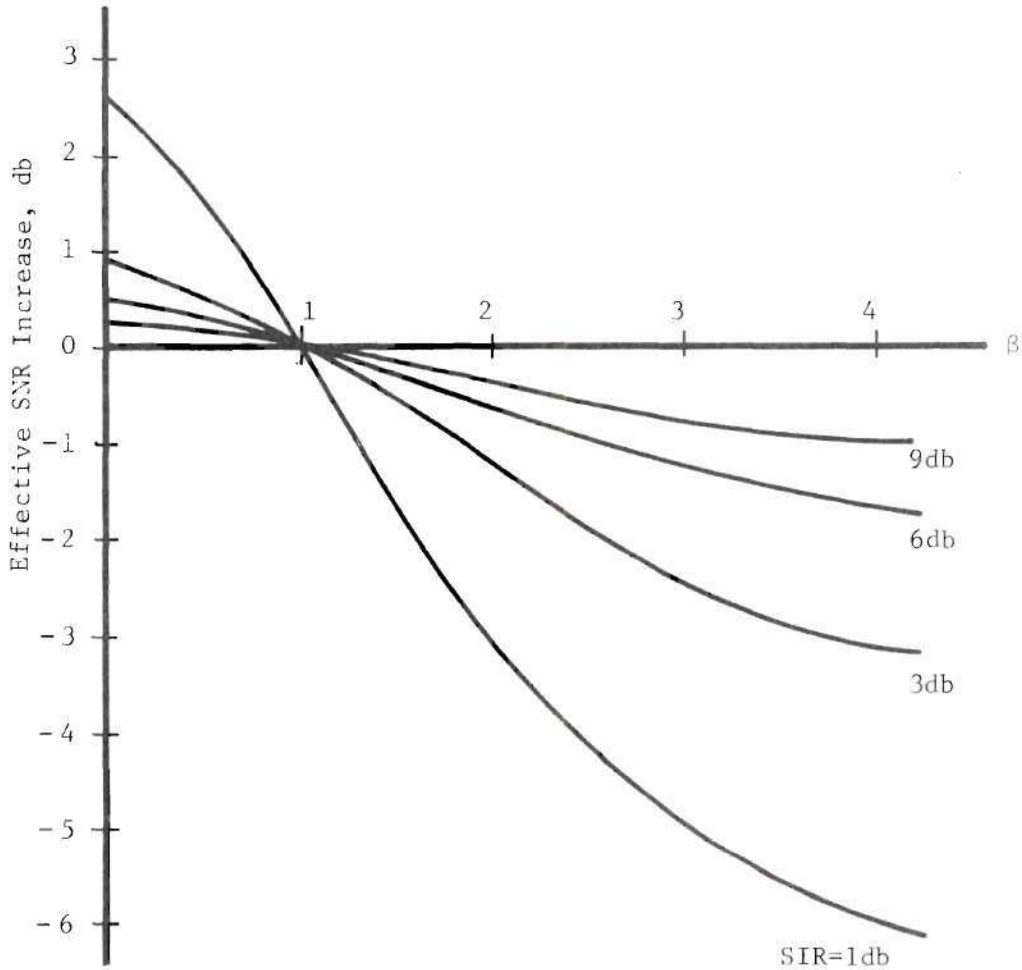

Figure 6. Effective SNR Increase In Figure 3:
Linear FM, Non-Unity Modulation Index

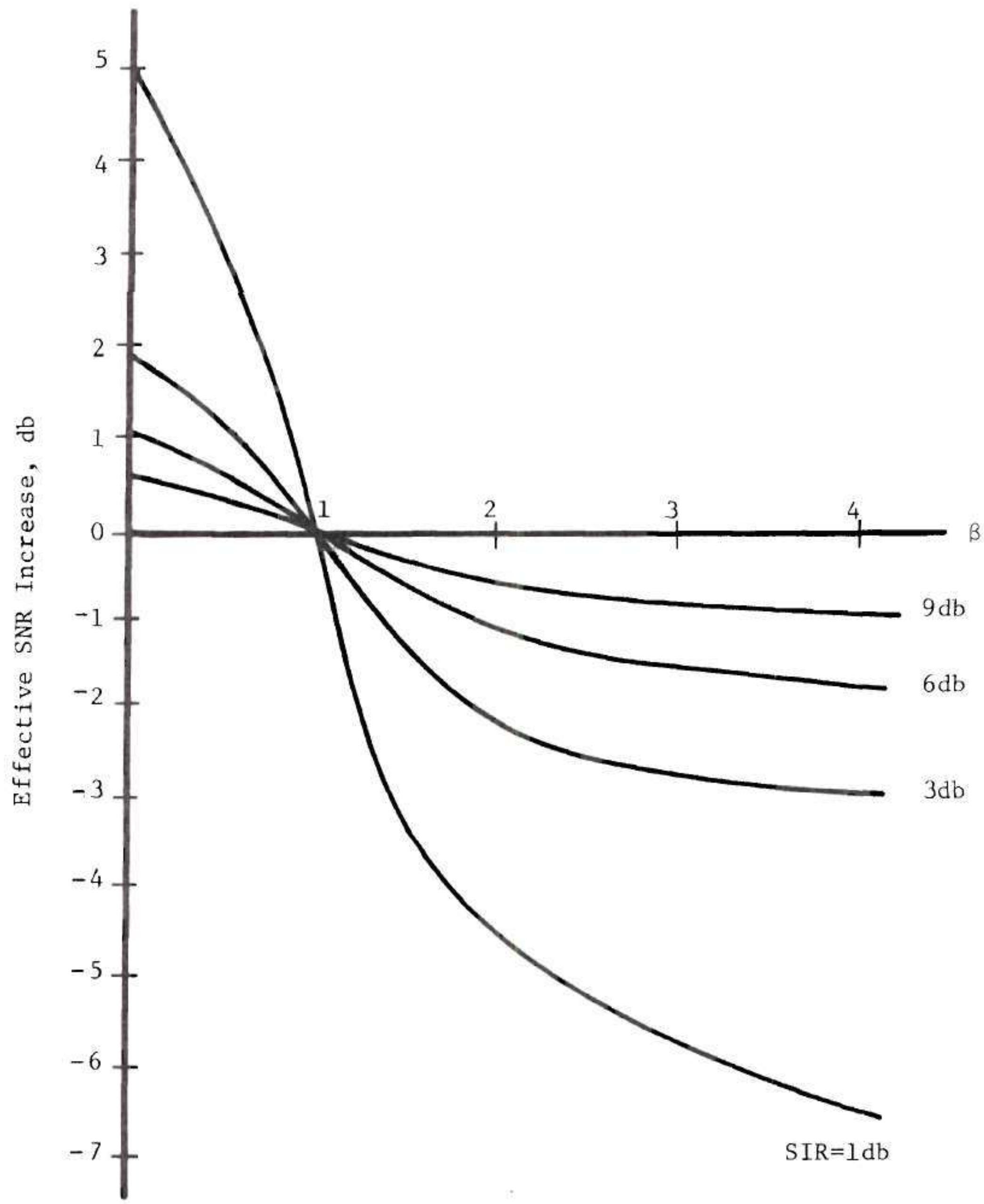

Figure 7. Effective SNR Increase in Figure 4: Sinusoidal FM, Non-Unity Modulation Index

A useful and alternate depiction of this inherent superiority is illustrated in Figure 8, which displays the variation in error rate for both types of interference as a function of modulation index, with SIR as a parameter. In this illustration, the SNR is held at a fixed value of 10db, for which the Gaussian-noise-only floor is shown. It can again be observed that CW interference induces the maximum error rate and that linear FM is superior to sinusoidal FM for any like values of β and SIR. In addition, Figure 8 can be used to determine the loss in effective SIR incurred by an increase of modulation index. For example, if a CW jamming source operating at any level of SIR changes to linear FM barrage jamming with $\beta = 3$, the jamming power must be increased by approximately 3db to maintain the same error rate. For sinusoidal FM barrage jamming, a 3db increase is required sooner, at approximately $\beta = 2$.

Pr(err): Bandpass Filter Modification

An analysis of system susceptibility for bandpass filters other than the first-order Butterworth has been made. Since the FM interference is wideband with respect to the bandpass filter in the barrage jamming mode, it was expected that the order and type of the filter would have a bearing on the resulting system error. The types of filters considered and their characteristics are detailed in a previous section of this chapter concerning computational methods.

The reference case of cochannel FM interference with unity modulation index was again analyzed initially. Since this is narrowband FM with the interfering power concentrated in the cochannel and

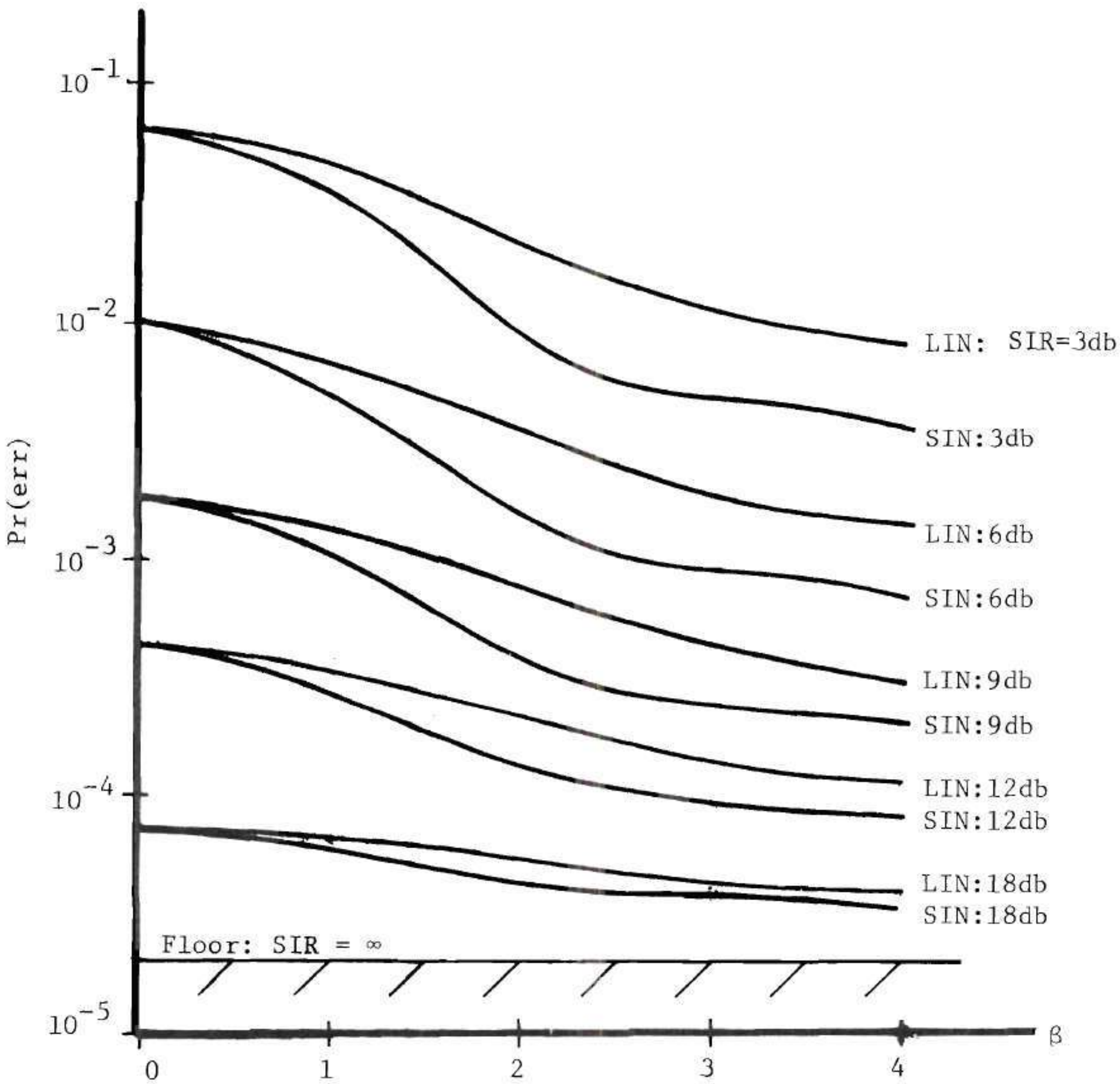

Figure 8. Cochannel Error Rate vs Modulation Index: Linear and Sinusoidal FM, SNR=10db

first sideband components, it was expected that the effect of the increased attenuation of the higher order filters beyond these components would be negligible. In Figure 9 the error curve for first-order Butterworth filtering (BW1) with SIR = 3db is again displayed. It was found that the curves for the second and fourth-order Butterworth filters are within only 1db of SNR of that shown for the first-order Butterworth. Of interest, however, was the fact that the shift is to the right, i.e., increasing attenuation of the higher order FM components results in a marginal increase in error rate for the system, in contrast to what one might expect. Chebyshev filtering was found to be of an identical nature. The ideal filtering case with $A(f_{co}) = .707$ also displayed similar characteristics, and generated the maximum curve shift, less than 1db to the right, of all filters with $A(f_{co}) = .707$. This curve, I (.7), is also shown in Figure 9. Thus the BW1 curve and I(.7) curve serve to tightly bound the error rates resulting from increased filter order. Furthermore, the SNR spread between the two curves has been found to decrease with increasing SIR, in a manner similar to that observed in a previous discussion concerning variation of the modulation index. The bounding thus becomes even tighter for higher SIR, so that the Pr(err) curves of Figure 3 can be used, for unity β, as excellent approximations for higher order filters and for SIR levels greater than the 3db level of Figure 9.

It is known [17] [18] that transmission of an FM signal through any bandlimited filter results in amplitude modulation, i.e., non-constancy of the FM envelope. Any attenuation in the amplitudes of the

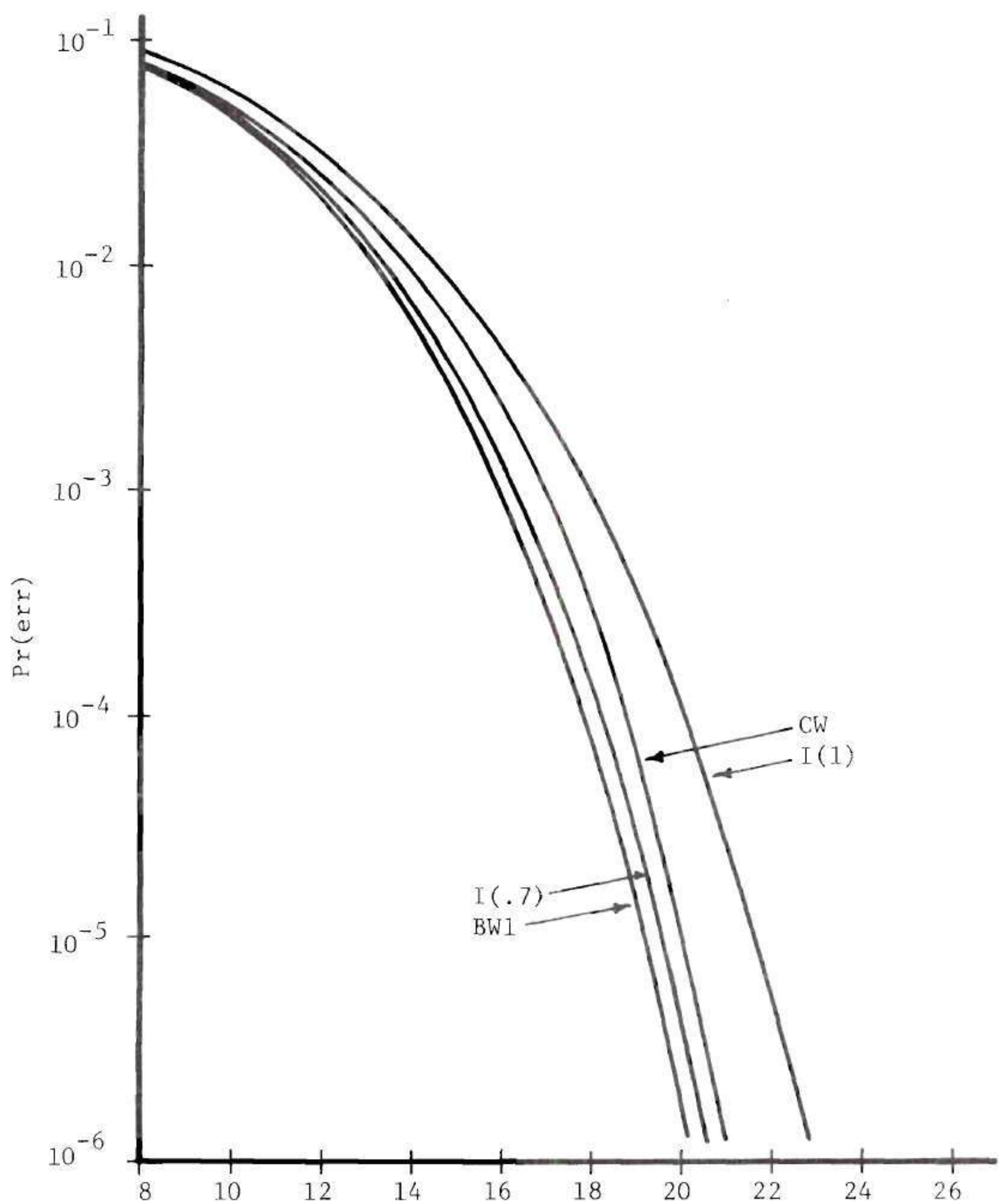

Figure 9. Cochannel Error Rate vs SNR and Filter Type: Linear FM, $\beta=1$, SIR=3db

sinusoidal components of the FM signal by the bandpass filters thus induces a variation in the output envelope. The Q-function is inherently non-linear, as indicated in Figure 10 from which it is apparent that the error rate increases exponentially with the argument α. The fact that the filtered FM envelope appears in its argument, in conjunction with the preceding results, suggest that the increased envelope variation engendered by higher order filtering can in certain cases offset the reduction in amplitude of the higher order FM components.

The error curve resulting from CW interference is also shown in Figure 9. It can be observed that for unity modulation index the previously mentioned error curves move toward the CW curve but are bounded by the curve I(.7), which is only slightly lower than that of the CW case. The question then arises as to whether there exists combinations of cochannel FM interferences and bandpass filters which generate more system error than that found for the CW case. From a jamming viewpoint, is there a cochannel FM interference which yields better jamming than the heretofore uniformly superior CW mode? Or alternately from a receiver viewpoint, given an unknown cochannel jamming interference, will increased order of filtering result in decreased performance, instead of improving it?

The answer is affirmative, as curve I(1) of Figure 9 illustrates. This curve is obtained by letting the first cochannel and sideband components of the FM interference fall within the unity attenuation range of the ideal filter, and attenuating to zero all others. This filter would thus be the limiting case for high-order filtering in

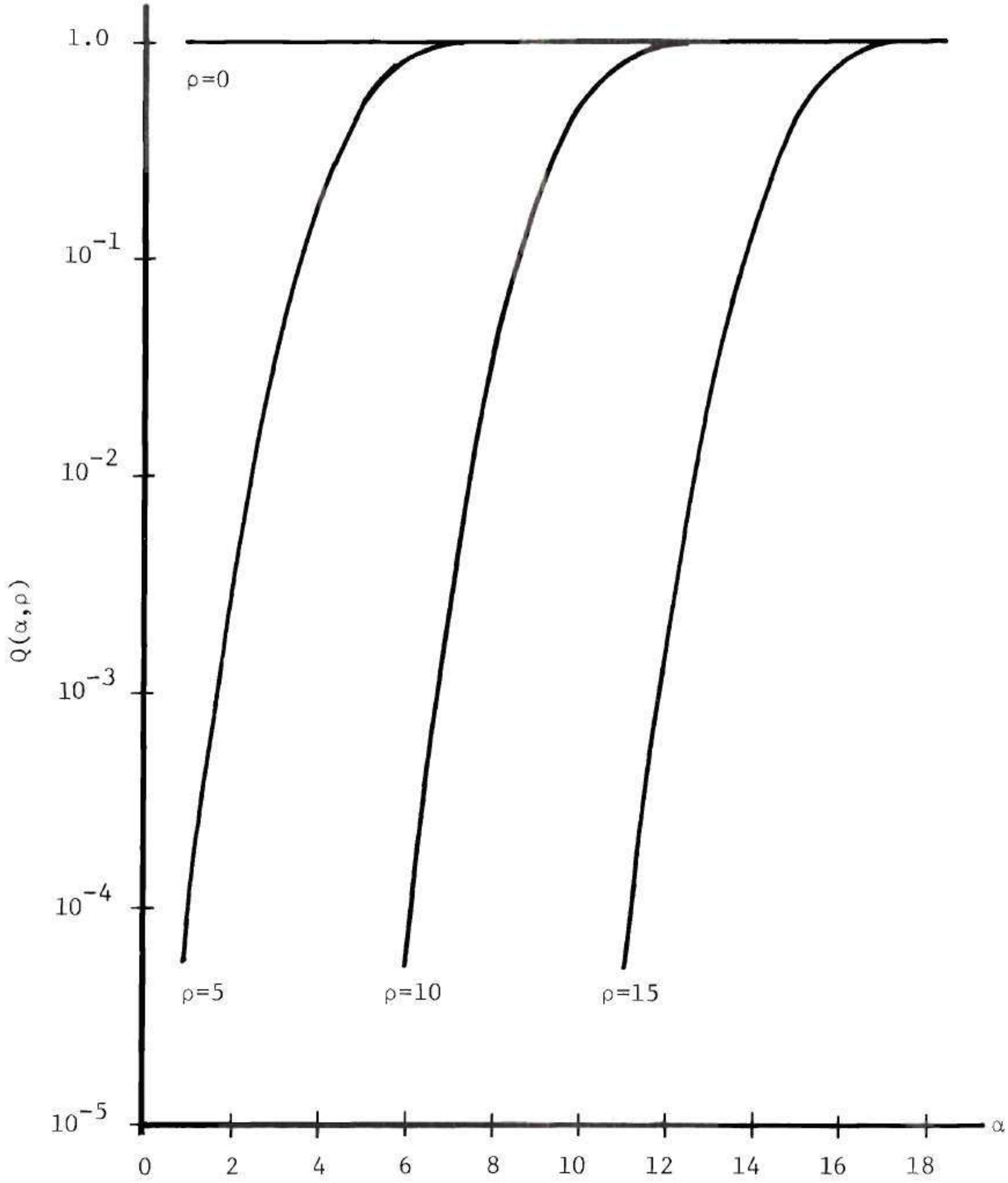

Figure 10. $Q(\alpha,\rho)$

which only the initial FM sideband components are within the filter passband. It is evident that an increase in jamming efficiency (or decrease in system performance) will occur, the loss in effective SNR ranging up to approximately 2db at the higher SNR levels.

Similar curves for $\beta = 2$ are presented in Figure 11. As before, the BW1 and I(.7) curves have been found to bound the curves obtained for the higher order, but non-ideal, filter types considered. In addition, the error rate for the I(1) filter has increased dramatically over that of the CW case, ranging to 8db of SNR for high SNR levels. Calculation of the power and envelope of the filtered linear FM via the available envelope subroutine demonstrated that whereas the output power level of the FM interference is reduced to 87% of its input level, its envelope excursion now ranges from 60% to 130% of its CW input level, i.e., severe amplitude modulation has occurred via high-order filtering and the system error rate increases accordingly. Furthermore, this value of β has been found to generate maximum error in the DPSK system.

Analysis of comparative data for cochannel sinusoidal FM has demonstrated that a completely similar trend for sinusoidal FM is not available. Sinusoidal FM has been found to be considerably inferior to either CW or linear FM for higher order filtering and for numerous values of modulation index. It is believed that the repeatedly demonstrated superiority of linear cochannel FM to sinusoidal cochannel FM resides in the more uniform distribution of power in linear FM throughout the bandwidth [18].

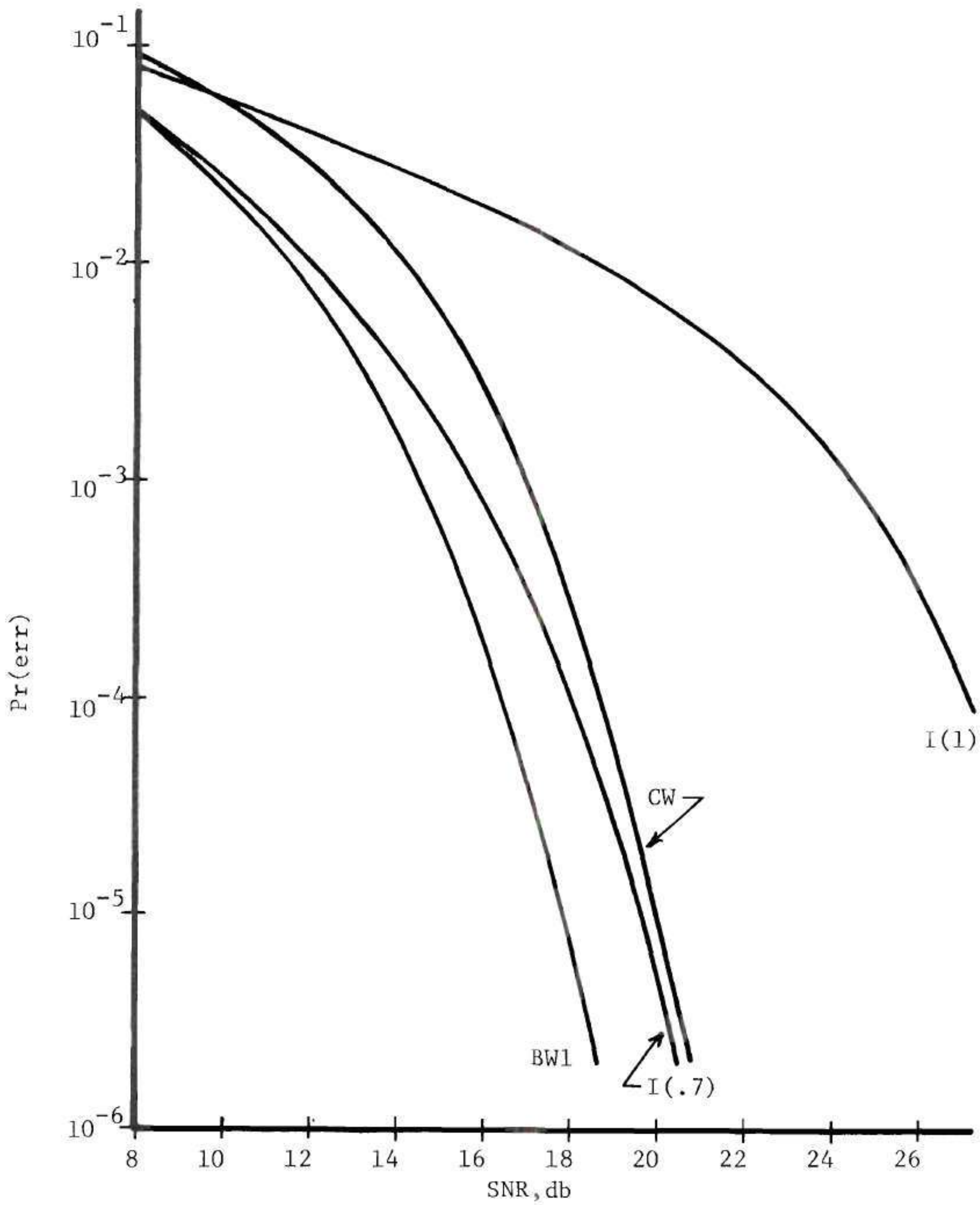

Figure 11. Cochannel Error Rate vs SNR and Filter Type: Linear FM, β=2, SIR = 3db

The conclusions that Chebyshev filtering offers no significant advantage over Butterworth filtering of the same order, and that higher-order filtering of either type can result in system degradation are of considerable interest, since parallel results have been found in an analysis of degradation due to bandpass filtering in phase-shift-keyed systems [29].

Adjacent Channel Interference

Analyses similar to those discussed for the case of cochannel FM interference have been conducted for the adjacent channel case, i.e., when the interfering carrier frequency w_j is centered in an adjacent channel. In this context, the adjacent channel is considered to define the bandwidth:

$$w_c + (2\pi/T_b) < w < w_c + (6\pi/T_b)$$

so that the interfering carrier frequency is

$$w_j = w_c + (4\pi/T_b)$$

The primary concern in this area has been the determination of system susceptibility, particularly with respect to that for cochannel interferences, i.e., whether a relative gain or loss in jamming efficiency can occur, and under what conditions. In addition, a comparison of system error rates for the different bandpass filtering techniques has been made.

Pr(err): First-Order Butterworth Filtering

Linear FM. For purposes of adjacent versus cochannel comparisons, the error rates for the cochannel case are displayed in an alternate form in Figure 12. The error rate is plotted as a function of SIR and SNR is now a parameter. It can again be observed that the system is quite susceptible for high jamming levels, i.e., the error rate becomes excessive at low SIR levels, regardless of system SNR. For given SNR, each curve of the error family must asymptotically approach the error rate for the Gaussian-noise-only case of equation (2-2) as the SIR becomes infinite.

As a result of extensive comparative calculations, it has been determined that the linear FM jamming source will suffer a loss in effectiveness if it is not cochannel with the signal. The loss is present regardless of any attempt to recover effectiveness via use of a barrage jamming mode. The consolidated results are shown in Figure 13, which specifies the degradation of jamming efficiency relative to the cochannel curves of Figure 12.

Spot versus Barrage Jamming. The CW or spot jamming mode, for which the modulation index is zero in Figure 13, suffers an immediate 7db loss as a direct consequence of the attenuation of the interfering carrier by the bandpass filter. As the modulation index is increased and the jamming source converts to barrage jamming, the degradation is lessened, the amount of the decrease dependent on the SNR of the system. The optimum modulation index occurs approximately at $\beta = 3.5$,

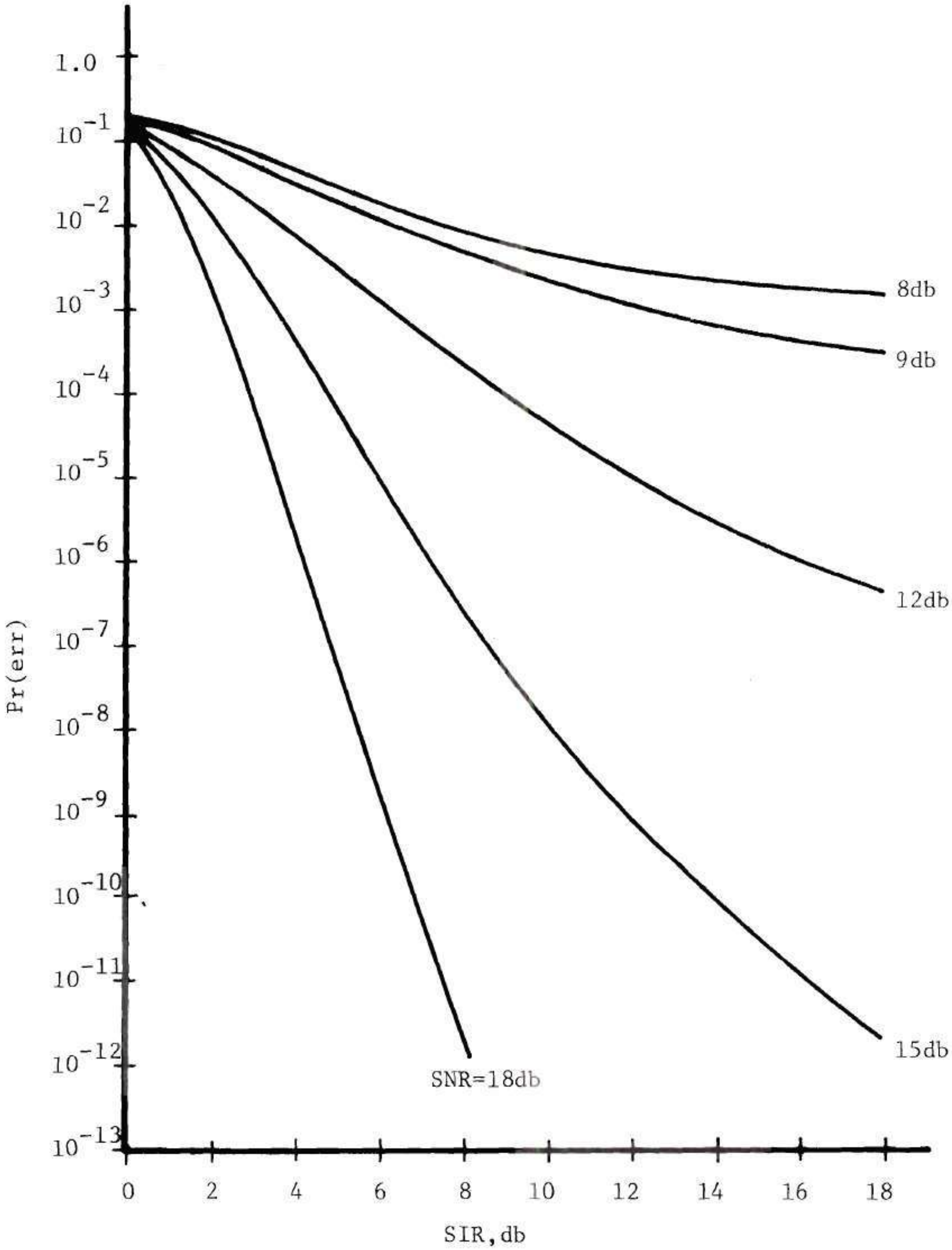

Figure 12. Cochannel Error Rate vs SIR and SNR: Linear FM, $\beta=1$

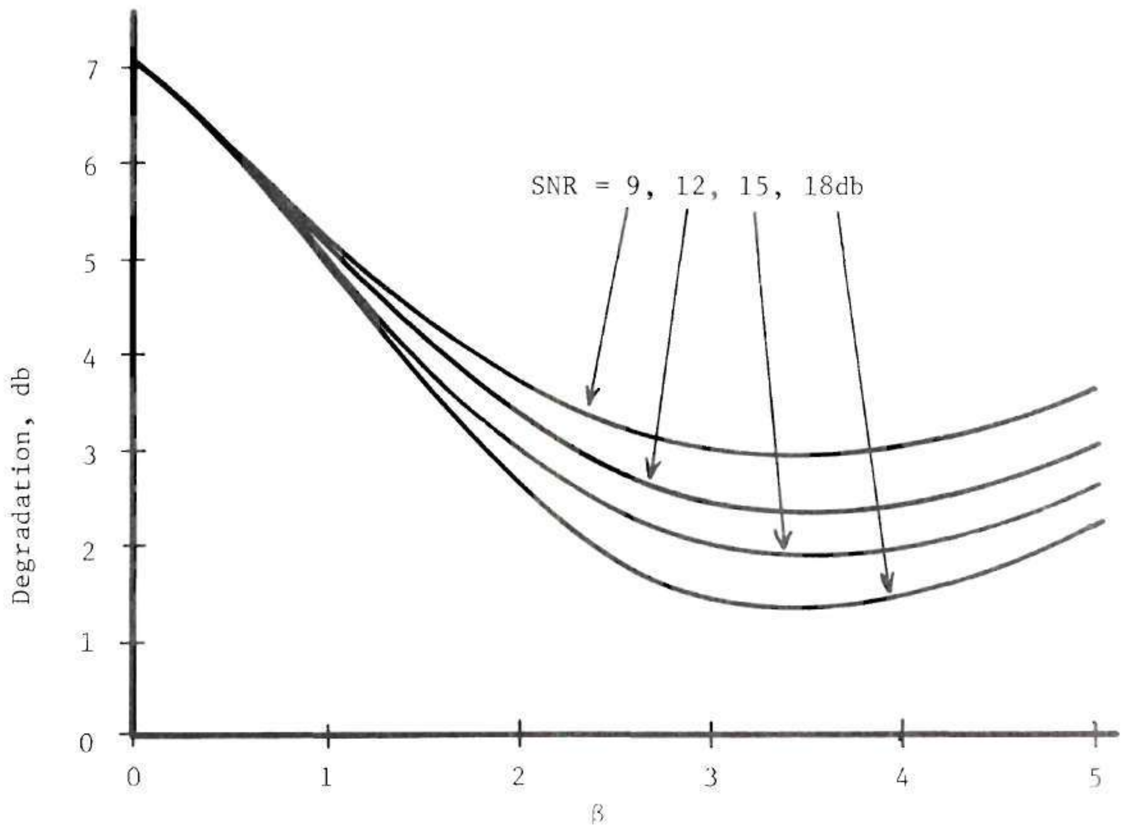

Figure 13. Degradation of Jamming Efficiency for Adjacent Channel Interference: Linear FM

for which degradation with respect to the cochannel case is still present, but minimum. Higher modulation indices only result in increased degradation of jamming efficiency. Adjacent channel interference is thus inferior to narrowband cochannel interference regardless of the barrage mode chosen by the jamming source.

The curves of Figure 13 can be used in conjunction with Figure 12 to determine similar error curves for the adjacent channel case. For specified values of SNR and modulation index, the degradation in jammer power can be obtained from Figure 13. The associated reference curve of Figure 12 is then shifted to the left by this amount to obtain the adjacent channel error rate at the same SNR and for the specified modulation index.

Sinusoidal FM. The use of sinusoidal FM by the adjacent channel source also results in a degradation of jamming efficiency, regardless of modulation index. In comparison with the linear FM mode, sinusoidal FM is again inferior except for a negligible improvement of less than 1db for modulation indices up to approximately $\beta = 2$, beyond which the linear FM mode is dominant. Considering the previously discussed superiority of linear FM for cochannel interference, linear FM is thus an overall superior jamming mode irrespective of the location of the interfering carrier.

Pr(err): Bandpass Filter Modification

The use of higher-order filtering has been found to decrease the system error rate from the levels obtained for a first-order Butterworth filter. In contrast to the cochannel case, an improvement in system operation is thus obtained by increased filtering, as a direct result of the increased attenuation of the adjacent channel interference. The improvement in system error rate is maximum for lower values of modulation index, for which case the interfering power spectrum is predominantly outside the filter passband. As the modulation index increases and the power spectrum spreads into the signal channel, the error rate increases but does not attain the level for first-order filtering.

Figure 14 is an illustrative example of these characteristics. The system SNR is chosen as 12db with an SIR of 6db. The error curve for first-order Butterworth filtering is maximum and rises steadily with modulation index from the error rate obtained by CW jamming, peaking at approximately $3x10^{-3}$ for a modulation index of 3.5, which is the optimum jammer choice, as noted previously from Figure 13.

The error rates obtained for the higher order filters can be observed to fall below the BW1 curve, the margin increasing as the jammer approaches the CW spot mode. As for the cochannel case, the I(.7) filter has been found to bound the error rate obtainable, however for the adjacent channel mode, the bound is a lower one. The fourth-order Butterworth and Chebyshev filters, yield almost identical error rates as for the ideal filter I(.7), thus only the curve for I(.7) is indicated. As in the cochannel case, no significant difference between Butterworth or Chebyshev filtering has been found.

The error rate using the ideal filter I(1), as shown in Figure 14,

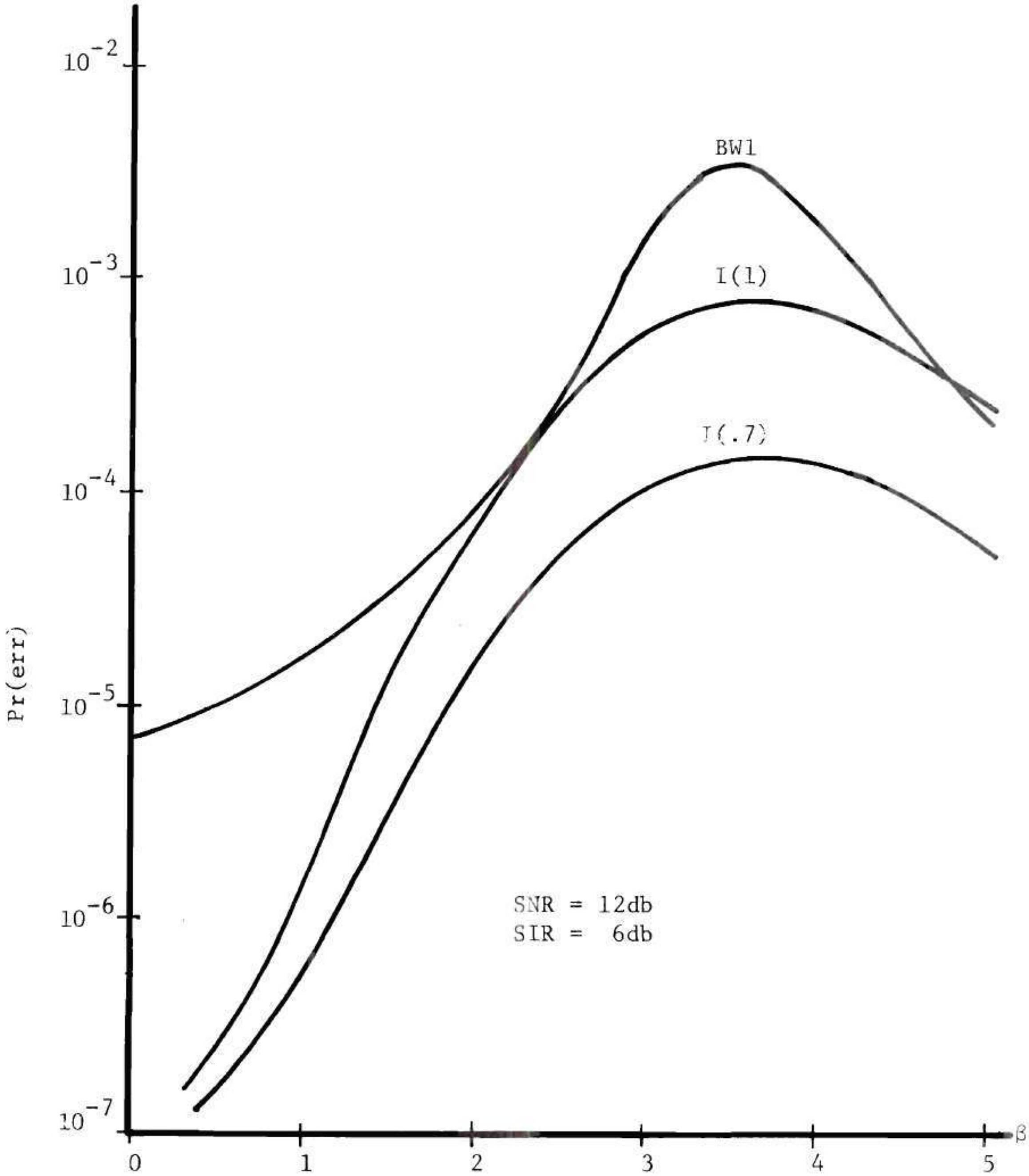

Figure 14. Adjacent Channel Error Rate vs Filter Type and Modulation Index: Linear FM

is again higher than those obtained by using the fourth-order or I(.7) filters. This increase is directly related to the increase in filter output power of the band-edge components of the jamming signal. Even with this extra inclusion, the system error rate obtained is lower than that occurring when a first-order Butterworth filter is employed.

Conclusions

CW versus FM Jamming

From a jamming viewpoint, the choice of whether to use CW spot jamming or some type of FM barrage jamming must ultimately depend on the knowledge the interfering source has with respect to the operating parameters of the DPSK system, as indicated in Table 1. If no information concerning the DPSK system is available other than a general knowledge of the approximate location of the signal channel, wideband barrage jamming is the only feasible choice for the jamming source, and a considerable loss in jamming efficiency will ensue. If however, knowledge of only the carrier frequency w_c has been obtained, cochannel CW interference is best, since any spreading of the interfering power spectrum via FM barrage jamming can result in a decreased error rate as compared to the CW case. If in addition the DPSK system bandwidth ($2f_{co}$) is known, cochannel, narrowband ($\beta = 2$) linear FM jamming with a modulation frequency f_m slightly less than or equal to f_{co} is preferable, since if the DPSK system is employing low-order filtering the jamming efficiency will approximate that of the CW case, whereas much increased error rates are obtainable if high-order

Table 1. Summary of Best Jammer/System Options

		Jamming Type	Filter Type*
A.	Unknown signal carrier f_c	linear FM barrage	--
B.	Known signal carrier frequency	CW cochannel	--
C.	Known signal carrier and system bandwidth $2f_{co}$	linear FM cochannel $f_m \leq f_{co}$ $\beta = 2$	--
D.	Unknown FM interference	--	low-order
E.	Narrowband FM Cochannel Interference	--	1st-order
F.	Wideband FM Barrage Interference	--	high-order

*consistent with signal constraints

filtering is being employed.

Linear versus Sinusoidal FM

If the jammer elects to use an FM interference, linear modulation is preferable to sinusoidal modulation. The underlying reason for the superiority of linear FM appears to reside in its more uniform distribution of power within the power spectrum of the interfering signal. For cochannel interference, linear FM is absolutely superior, whereas for the non-cochannel modes, linear FM is, in an average sense, superior over a wider range of modulation index. Furthermore, the relative superiority of linear modulation increases with increased jamming power.

Low-order versus High-order Bandpass Filtering

From the DPSK system viewpoint, the question of what order is best for the bandpass filter is a complex one, being dependent not only on noise levels and adjacent channel signal discrimination, but also on the type of deliberate interference that may be present. Returning again to Table 1, if the system design is required to be fixed and no assumption made as to type of interference, the use of the lowest order filtering consistent with proper system operation is indicated, as higher order filtering increases not only the likelihood of intersymbol interference [29], but also the likelihood of increased jamming efficiency in the cochannel case.

If the DPSK system is restricted to operating with a specified carrier frequency and bandwidth, but interrupt monitoring of the FM source is available, then a possible countermeasure exists. If the interference is cochannel and narrowband with most of the interfering power residing in the signal channel, use of 1st-order filtering is best, as high-order filtering can only serve to accentuate the problem. If however, the FM interference is of a barrage jamming mode, then a bandpass filter with highest order is indicated, consistent with other signal constraints. Furthermore, no significant difference in system susceptibility has been found to exist between the Chebyshev and Butterworth filters of identical order, thus no preference has been denoted in Table 1.

Recommendations

The development that linear FM interference can often cause larger error rates in a DPSK system than can CW interference of the same power, implies that there may exist other modulating signals for FM interference which are superior to both. The modulating signal should likely be such that the envelope of the interference at the bandpass filter output contains a maximum excursion with respect to the envelope of the CW interference. Due to the non-linearity of the Q-function and the averaging over a signaling interval, it appears that peak envelope excursion (peak power) is weighted more than average power. Further investigation is plausible in this area.

Furthermore, it is believed that the techniques presented herein can be extended to analyze the case of mutual interference between similar users, either adjacent or cochannel to each other. The reason for this supposition resides in the freedom of choice concerning the modulation waveform m(t). For example, mutual interference between similar PSK systems could possibly be modeled by choosing m(t) as a periodic impulse train. For NCFSK systems, m(t) could possibly be chosen as a periodic pulse train. As in this work, the effects of random phase and random synchronization would of course need to be incorporated.

APPENDIX I

FOURIER COEFFICIENTS OF LINEAR FM INTERFERENCE

Directly from Figure 2 the analytical expression for the linear modulation m(t) is:

$$m(t) = \begin{cases} m_1(t) = 2\Delta w(t - t_o + (T_m/2))/T_m & 0 \le t < t_o \quad \text{(A1-1)} \\ m_2(t) = 2\Delta w(t - t_o - (T_m/2))/T_m & t_o \le t < T_m \quad \text{(A1-2)} \end{cases}$$

Defining the modulation index β in the usual manner as

$$\beta = \Delta w/w_m$$

then

$$\Delta w = \beta w_m$$

With

$$T_m = 2\pi/w_m$$

then

$$m_1(t) = (\beta w_m^2/\pi)\ (t - t_o + (T_m/2)) \qquad 0 \le t < t_o \tag{A1-3}$$

$$m_2(t) = (\beta w_m^2/\pi)\ (t - t_o - (T_m/2)) \qquad t_o \le t < T_m \tag{A1-4}$$

From equation (2-20) we have

$$a_n = (1/T_m) \int_0^{T_m} \exp[j \int_0^t m(t_1)dt_1]\exp[-jnw_m t]dt \tag{A1-5}$$

$$= (1/T_m) \int_0^{t_o} \exp[j \int_0^t m_1(t_1)dt_1]\exp[-jnw_m t]dt \tag{A1-6}$$

$$+ (1/Tm) \int_{t_o}^{T_m} \exp[j \int_0^{t_o} m_1(t_1)dt_1 + j \int_{t_o}^{t} m_2(t_1)dt_1]$$

$$\times \exp[-jnw_m t]dt$$

Since $t > t_o$ in the second integral, the inner integral must be evaluated over the ranges indicated, with the proper $m_i(t)$. Since

$$\int_0^t m_1(t_1)dt_1 = \int_0^t (\beta w_m^2/\pi)\ (t_1 - t_o + (T_m/2))dt_1 \tag{A1-7}$$

$$= (\beta w_m^2/2\pi)\ (t^2 + (T_m - 2t_o)t)$$

then in the second integral:

$$\int_0^{t_o} m_1(t_1)dt_1 = (\beta w_m^2/2\pi)\ (T_m t_o - t_o^2) \tag{A1-8}$$

In addition

$$\int_{t_o}^{t} m_2(t_1)dt_1 = \int_{t_o}^{t} (\beta w_m^2/\pi)\ (t_1 - t_o - (T_m/2))dt_1 \tag{A1-9}$$

$$= (\beta w_m^2/2\pi)[(t^2 - (T_m + 2t_o)t + t_o^2 + T_m t_o]$$

Substituting (A1-7), (A1-8), and (A1-9) into (A1-6) we have:

$$a_n = (1/T_m) \int_0^{t_o} \exp[j\beta w_m^2 t^2/2\pi] \exp\{j[(\beta-n)w_m - (\beta w_m^2 t_o/\pi)]t\}dt \tag{A1-10}$$

$$+ (1/T_m)\exp[j2\beta w_m t_o] \int_{t_o}^{T} \exp[j\beta w_m^2 t^2/2\pi]$$

$$\times \exp\{-j[(\beta + n)w_m + (\beta w_m^2 t_o/\pi)]t\}dt$$

The integrals in (A1-10) are more easily evaluated by "completing the square" in each integrand, yielding, after considerable manipulation:

$$a_n = \exp[-j\pi(\beta-n)^2/2\beta]\ \exp[j(\beta-n)w_m t_o] \cdot \exp[-j\beta w_m^2 t_o^2/2\pi] \tag{A1-11}$$

$$\times (1/T_m) \int_0^{t_o} \exp\{j(\beta w_m^2/2\pi)[t + (\pi/\beta w_m^2)(\beta w_m - n w_m) - (\beta w_m^2 t_o^2/\pi)]^2\}dt$$

$$+ \exp[j2\beta w_m t_o]\exp[-j\pi(\beta+n)^2/2\beta]\exp[-j(\beta+n)w_m t_o] \times \exp[-j\beta w_m^2 t_o^2/2\pi]$$

$$\times (1/T_m) \int_{t_o}^{T_m} \exp\{j(\beta w_m^2/2\pi)[t-(\pi/\beta w_m^2)(\beta w_m+nw_m+(\beta w_m^2 t_o/\pi)]^2\}dt$$

Since

$$\exp[j2\beta w_m t_o]\exp[-j(\beta+n)w_m t_o] = \exp[j(\beta-n)w_m t_o]$$

$$\exp[-j\pi(\beta-n)^2/2\beta] = \exp[-j\pi(\beta+n)^2/2\beta]$$

$$= \exp[-j\pi(n^2+\beta^2)/2\beta] \exp[-jn\pi]$$

$$= (-1)^n \exp[-j\pi(n^2+\beta^2)/2\beta]$$

then the coefficients of each integral in (A1-11) are identical so that

$$a_n = (-1)^n \exp[-j\pi(n^2+\beta^2)/2\beta] \exp[j(\beta-n)w_m t_o] \qquad \text{(A1-12)}$$

$$\times \exp[-j\beta w_m^2 t_o^2/2\pi]$$

$$\times (1/T_m)\Bigg[\int_o^{t_o} \exp\{j(\beta w_m^2/2\pi)[t+(\pi/\beta w_m^2)(\beta w_m - nw_m$$

$$- (\beta w_m^2 t_o/\pi)]^2 dt + \int_{t_o}^{T_m} \exp\{j(\beta w_m^2/2\pi)$$

$$\times [t-(\pi/\beta w_m^2)(\beta w_m + (\beta w_m^2 t_o/\pi)+nw_m]^2\}dt\Bigg]$$

Since

$$\int_o^x \exp[j\pi\tau^2/2]d\tau = \int_o^x \cos(\pi\tau_1^2/2)d\tau_1 \tag{A1-13}$$

$$+ j \int_o^x \sin(\pi\tau_2^2/2)d\tau_2$$

$$= C(x) + j\,S(x)$$

where C(x) and S(x) are the Fresnel integrals [30], we now form the Fresnel integrals by change of variable, i.e., we set

$$(\pi\tau_1^2/2) = (\beta w_m^2/2\pi)[t+(\pi/\beta w_m^2)(\beta w_m - n w_m - (\beta w_m^2 t_o/\pi)]^2 \tag{A1-14}$$

$$(\pi\tau_2^2/2) = (\beta w_m^2/2\pi)[t-(\pi/\beta w_m^2)(\beta w_m + n w_m + (\beta w_m^2 t_o/\pi)]^2 \tag{A1-15}$$

Thus

$$\tau_1 = (\sqrt{\beta}\; w_m/\pi)[t + (\pi/\beta w_m)(\beta - n) - t_o] \tag{A1-16}$$

$$\tau_2 = (\sqrt{\beta}\; w_m/\pi)[t - (\pi/\beta w_m)(\beta + n) - t_o] \tag{A1-17}$$

$$d\tau_1 = (\sqrt{\beta}\; w_m/\pi)(dt) \tag{A1-18}$$

$$d\tau_2 = (\sqrt{\beta}\; w_m/\pi)(dt) \tag{A1-19}$$

Substituting the relations (A1-14) to (A1-19) and evaluating the new limits of integration via (A1-16) and (A1-17), the relation (A1-12) becomes:

$$a_n = (\Psi/2\sqrt{\beta})\left[\int_{a-b}^{a} \exp[j\pi\tau_1^2/2]d\tau_1 + \int_{d}^{a-b} \exp[j\pi\tau_2^2/2]d\tau_2\right] \qquad \text{(A1-20)}$$

$$= (\Psi/2\sqrt{\beta}) \int_{d}^{a} \exp[j\pi\tau^2/2]d\tau \qquad \text{(A1-21)}$$

in which the definitions

$$a = (\beta-n)/\sqrt{\beta} \qquad \text{(A1-22)}$$

$$b = \sqrt{\beta}\, w_m t_o/\pi \qquad \text{(A1-23)}$$

$$d = -(\beta+n)/\sqrt{\beta} \qquad \text{(A1-24)}$$

$$\Psi = (-1)^n \exp[-j\pi(n^2+\beta^2)/2\beta]\exp[j(\beta-n)w_m t_o] \qquad \text{(A1-25)}$$

$$\times \exp[-j\beta w_m^2 t_o^2/2\pi]$$

have been made for simplicity. Fresnel integrals are now obtained since

$$a_n = (\Psi/2\sqrt{\beta})\left[\int_{o}^{a} \exp[j\pi\tau^2/2]d\tau + \int_{d}^{o} \exp[j\pi\tau^2/2]d\tau\right]$$

$$= (\Psi/2\sqrt{\beta}) \left[\int_o^a \exp[j\pi\tau^2/2]d\tau - \int_o^d \exp[j\pi\tau^2/2]d\tau \right]$$

$$= (\Psi/2\sqrt{\beta}) \left[C(a) + jS(a) - C(d) - jS(d) \right]$$

$$= (\Psi/2\sqrt{\beta}) \left[C(a) + C(-d) + jS(a) + jS(-d) \right] \tag{A1-26}$$

Note that in obtaining (A1-26) use has been made of the relations [30]

$$C(x) = -C(-x)$$

$$S(x) = -S(-x)$$

Equation (A1-26), with the substitutions (A1-22), (A1-24), and (A1-25), constitutes the general solution for the set of complex Fourier coefficients. The magnitude $|a_n|$ and phase ϕ_n of the coefficients are immediately evident, i.e.,

$$|a_n| = (1/\sqrt{2\beta}) \left[[C(a) + C(-d)]^2 + [S(a) + S(-d)]^2 \right]^{\frac{1}{2}} \tag{A1-27}$$

$$\phi_n = \Psi + \arctan \{[S(a) + S(-d)]/[C(a) + C(-d)]\} \tag{A1-28}$$

such that

$$a_n = |a_n| \exp [j\phi_n]$$

as specified in (2-21). The phase ϕ_n can be further expressed in terms of the uniform random variable, x, as defined in the relation (3-84), i.e.:

$$\phi_n = (-1)^{n+1} \, [\beta x^2 + (n-\beta)x + (\pi(n^2 + \beta^2)/2\beta)]$$

$$+ \arctan \{[S(a) + S(-d)]/[C(a) + C(-d)]\} \qquad \text{(A1-29)}$$

The Fourier Coefficients for sinusoidal FM are cited and the derivations widely available in the literature [17] [18] [19].

APPENDIX II

DERIVATION OF $Pr(err|H_1)$

Beginning with the error relationship (3-42):

$$Pr(err|H_1) = Pr(R_1 < R_o) \tag{A2-1}$$

then in terms of the conditional density functions $f(r_1|H_1)$ and $f(r_o|H_1)$ we have:

$$Pr(err|H_1) = \int_{r_1=0}^{\infty} \int_{r_o=r_1}^{\infty} f(r_o|H_1) f(r_1|H_1) dr_o dr_1 \tag{A2-2}$$

Since R_o and R_1 are themselves functions of the random variables Φ, U_1, V_1, U_2, and V_2, the above density functions must be obtained via a transformation of variables [22].

Consider the determination of $f(r_o|H_1)$. First we define in R_o the set of "old" random variables:

$$X_1 = U_2 \tag{A2-3}$$

$$X_2 = V_2 \tag{A2-4}$$

$$X_3 = \Phi \tag{A2-5}$$

and a "new" set of random variables:

$$Z_1 = R_o$$

$$= \{[D\sin\Phi - C\cos\Phi + V_2]^2 + [D\cos\Phi + C\sin\Phi + U_2]^2\}^{\frac{1}{2}} \quad \text{(A2-6)}$$

$$Z_2 = \arctan\left[\frac{D\sin\Phi - C\cos\Phi + V_2}{D\cos\Phi + C\sin\Phi + U_2}\right] \quad \text{(A2-7)}$$

$$Z_3 = \Phi \quad \text{(A2-8)}$$

since under the assumption "H_1 true", the A_s terms in the right hand side of (3-40) cancel, thus yielding the result (A2-6).

In terms of the old set, the new set can be expressed as

$$Z_1 = R_o \quad \text{(A2-9)}$$

$$Z_2 = \arctan\left[\frac{D\sin X_3 - C\cos X_3 + X_2}{D\cos X_3 + C\sin X_3 + X_1}\right] \quad \text{(A2-10)}$$

$$Z_3 = X_3 \quad \text{(A2-11)}$$

From the definitions (A2-6) to (A2-8) it follows that

$$Z_1\cos Z_2 = D\cos X_3 + C\sin X_3 + X_1 \quad \text{(A2-12)}$$

$$Z_1 \sin Z_2 = D \sin X_3 - C \cos X_3 + X_2 \tag{A2-13}$$

$$Z_3 = X_3 \tag{A2-14}$$

Substitution of (A2-14) into (A2-12) and (A2-13) yields the solution of the old set in terms of the new set, i.e.:

$$X_1 = Z_1 \cos Z_2 - D \cos Z_3 - C \sin Z_3 \tag{A2-15}$$

$$X_2 = Z_1 \sin Z_2 - D \sin Z_3 + C \cos Z_3 \tag{A2-16}$$

$$X_3 = Z_3 \tag{A2-17}$$

The joint density function $f(z_1, z_2, z_3)$ of the new set of random variables is [22]

$$f(z_1, z_2, z_3) = f(x_1, x_2, x_3)\, J\left(\frac{x}{z} \right) \tag{A2-18}$$

where $f(x_1, x_2, x_3)$ is the joint density function of the old set of random variables and $J\left(\frac{x}{z} \right)$ is the Jacobian:

$$J\left(\frac{x}{z}\right) = \begin{vmatrix} \frac{\partial X_1}{\partial Z_1} & \frac{\partial X_2}{\partial Z_1} & \frac{\partial X_3}{\partial Z_1} \\ \frac{\partial X_1}{\partial Z_2} & \frac{\partial X_2}{\partial Z_2} & \frac{\partial X_3}{\partial Z_2} \\ \frac{\partial X_1}{\partial Z_3} & \frac{\partial X_2}{\partial Z_3} & \frac{\partial X_3}{\partial Z_3} \end{vmatrix} \tag{A2-19}$$

Obtaining the indicated partial derivatives of the set (A2-15) to (A2-17) and solving the determinant yields

$$J\left(\frac{x}{z}\right) = Z_1 \tag{A2-20}$$

The joint density function $f(x_1, x_2, x_3)$ is easily obtained by noting from previous discussions that x_1 and x_2 are independent Gaussian random variables and x_3 is an independent uniformly distributed random variable so that

$$f(x_1, x_2, x_3) = f(x_1)\, f(x_2)\, f(x_3) \tag{A2-21}$$

and

$$f(x_1) = (1/\sqrt{4\pi N}\ ) \exp[-x_1^2/4N] \qquad -\infty<x_1<\infty \tag{A2-22}$$

$$f(x_2) = (1/\sqrt{4\pi N}\ ) \exp[-x_2^2/4N] \qquad -\infty<x_2<\infty \tag{A2-23}$$

$$f(x_3) = 1/2\pi \qquad 0 \le x_3 < 2\pi \tag{A2-24}$$

Thus

$$f(x_1,\ x_2,\ x_3) = (1/8\pi^2 N) \exp[-(x_1{}^2 + x_2{}^2) \ /\ 4N] \tag{A2-25}$$

Using the solutions (A2-15) to (A2-17), the joint density function of the new set of random variables is then

$$f(z_1,z_2,z_3) = (z_1/8\pi^2 N) \tag{A2-26}$$

$$\times \exp\left\{ \frac{[z_1\cos z_2 - D\cos z_3 - C\sin z_3]^2 + [z_1\sin z_2 - D\sin z_3 + C\cos z_3]^2}{4N} \right\}$$

$$0 \le z_1 < \infty$$

$$0 \le z_2 < 2\pi$$

$$0 \le z_3 < 2\pi$$

Performing the indicated squaring yields, after considerable manipulation:

$$f(z_1,z_2,z_3) = (z_1/8\pi^2 N)\exp[-(z_1{}^2+C^2+D^2)/4N] \tag{A2-27}$$

$$\times \exp\left[\frac{z_1}{2N}(D\cos z_3+C\sin z_3)\cos z_2\right.$$

$$\left.+\ \frac{z_1}{2N}(D\sin z_3-C\cos z_3)\sin z_2\right]$$

$$0 \le z_1 < \infty$$

$$0 \le z_2, z_3 < 2\pi$$

Integrating over the range of Z_2:

$$f(z_1,z_3) = \int_o^{2\pi} f(z_1,z_2,z_3)dz_2 \tag{A2-28}$$

By use of the relation (3-49) we have:

$$f(z_1,z_3) = (z_1/4\pi N)\exp[-(z_1{}^2+C^2+D^2)/4N] \tag{A2-29}$$

$$\times\ I_o\left[\frac{z_1}{2N}\{(D\cos z_3+C\sin z_3)^2+(D\sin z_3-C\cos z_3)^2\}^{\frac{1}{2}}\right]$$

which after some manipulation, is:

$$f(z_1,z_3)=(z_1/4\pi N)\exp[-(z_1{}^2+C^2+D^2)/4N]I_o[z_1(C^2+D^2)^{\frac{1}{2}}/2N] \qquad \text{(A2-30)}$$

Defining

$$\gamma^2 = C^2 + D^2 \qquad \text{(A2-31)}$$

and noting that the variable Z_3 no longer appears in (A2-30), we have:

$$f(z_1) = \int_o^{2\pi} f(z_1,z_3)dz_3 \qquad \text{(A2-32)}$$

$$= 2\pi\ f(z_1,z_3)$$

Thus since $z_1 = R_o$ as defined previously, we have under hypothesis H_1 that

$$f(r_o|H_1) = (r_o/2N)\exp[-(r_o^2+\gamma^2)/4N]I_o[r_o\gamma/2N] \qquad \text{(A2-33)}$$

The calculation of $f(r_o|H_1)$ is now complete. The determination of $f(r_1|H_1)$ is obtained in a similar manner. Noting that under the hypothesis "H_1 true" the A_s terms in R_1 add in (3-40), then

$$R_1 = \{[2A_s + B\cos\Phi - C\sin\Phi + U_1]^2 \tag{A2-34}$$

$$+ [B\sin\Phi + C\cos\Phi \ + V_1]^2\}^{\frac{1}{2}}$$

Defining

$$X_1 = U_1 \tag{A2-35}$$

$$X_2 = V_2 \tag{A2-36}$$

$$X_3 = \Phi \tag{A2-37}$$

as the set of "old" random variables in R_1 and defining a "new" set of random variables as

$$Y_1 = R_1 \tag{A2-38}$$

$$Y_2 = \arctan\left[\frac{B\sin\Phi + C\cos\Phi \ + V_1}{2A_s + B\cos\Phi - C\sin\Phi + U_1}\right] \tag{A2-39}$$

$$Y_3 = \Phi \tag{A2-40}$$

then in terms of the "old" set:

$$Y_1 = R_1 \tag{A2-41}$$

$$Y_2 = \arctan\left[\frac{B\sin X_3 + C\cos X_3 + X_2}{2A_s + B\cos X_3 - C\sin X_3 + X_1}\right] \tag{A2-42}$$

$$Y_3 = X_3 \tag{A2-43}$$

From equation (A2-34), it then follows that

$$Y_1\cos Y_2 = 2A_s + B\cos X_3 - C\sin X_3 + X_1 \tag{A2-44}$$

$$Y_1\sin Y_2 = B\sin X_3 + C\cos X_3 + X_2 \tag{A2-45}$$

$$Y_3 = X_3 \tag{A2-46}$$

and noting that $X_3 = Y_3$, the solution of the old variables in terms of the new variables is:

$$X_1 = Y_1\cos Y_2 - 2A_s - B\cos Y_3 + C\sin Y_3 \tag{A2-47}$$

$$X_2 = Y_1\sin Y_2 - B\sin Y_3 - C\cos Y_3 \tag{A2-48}$$

$$X_3 = Y_3 \tag{A2-49}$$

The Jacobian of this transformation is

$$J\left(\frac{x}{y}\right) = Y_1 \tag{A2-50}$$

and again, since the random variables X_1, X_2, X_3 are independent:

$$f(x_1,x_2,x_3) = (1/8\pi^2 N)\ \exp[-(x_1^{\ 2} + x_2^{\ 2})/4N] \qquad \text{(A2-51)}$$

where now X_1 and X_2 are as given in (A2-47) and (A2-48). The joint density function of the new set of random variables is, via (A2-18):

$$f(y_1,y_2,y_3) = f(x_1,x_2,x_3)\ \cdot J\left(\frac{x}{y}\right) \qquad \text{(A2-52)}$$

$$= (y_1/8\pi^2 N)\ \exp[-(x_1^{\ 2} + x_2^{\ 2})/4N]$$

Substituting in the relations for X_1 and X_2 yields, after considerable manipulation:

$$f(y_1,y_2,y_3) = (y_1/8\pi^2 N)\exp[-(y_1^{\ 2} + \alpha_1^{\ 2}(y_3))/4N]\exp[y_1/2N] \qquad \text{(A2-53)}$$

$$\times\ \exp[(B\cos y_3 + 2A_s - C\sin y_3)\cos y_2 + (B\sin y_3 + C\cos y_3)\sin y_2]$$

where for simplification the parameter

$$\alpha_1^{\ 2}(y_3) = 4A_s^{\ 2} + B^2 + C^2 + 4A_s(B\cos y_3 - C\sin y_3) \qquad \text{(A2-54)}$$

has been defined.

Integrating over the range of the variable y_2 and noting that the integration again yields the Bessel function we have:

$$f(y_1,y_3) = (y_1/4\pi N)\ \exp[-(y_1^{\ 2} + \alpha_1^{\ 2}(y_3))/4N] \tag{A2-55}$$

$$\times \quad I_o[\frac{y_1}{2N}\{(B\cos y_3 + 2A_s - C\sin y_3)^2$$

$$+ (B\sin y_3 + C\cos y_3)^2\}^{\frac{1}{2}}]$$

Performing the indicated squaring and summing of terms in the argument of the Bessel function yields:

$$f(y_1,y_3) = (y_1/4\pi N)\ \exp[-(y_1^{\ 2} + \alpha_1^{\ 2}(y_3))/4N] \tag{A2-56}$$

$$\times\ I_o[y_1\alpha_1(y_3)/2N]$$

The probability density function for y_1 is then

$$f(y_1) = \int_o^{2\pi} f(y_1,y_3)\ dy_3 \tag{A2-57}$$

In this case the integrand is a quite complex function of y_3 in contrast to the solution for $f(z_1)$ in which the similar variable z_3 was not present in the integrand.

Since $Y_1 = R_1$ as defined previously, we have from (A2-56) and

(A2-57) that under hypothesis H_1:

$$f(r_1|H_1) = \int_0^{2\pi} (r_1/4\pi N)\exp[-(r_1^{\,2}+\alpha_1^{\,2}(y_3))/4N]I_o[y_1\alpha_1(y_3)/2N]dy_3 \quad \text{(A2-58)}$$

We are now in a position to substitute (A2-58) and (A2-33) into (A2-2), thus obtaining the error probability under hypothesis H_1:

$$\Pr(\text{err}|H_1) = \int_{r_1=0}^{\infty}\int_{r_o=r_1}^{\infty}\int_{y_3=0}^{2\pi} \{ \frac{r_o}{2N} \exp[- \frac{r_o^{\,2} + \gamma^2}{4N}] I_o[\frac{r_o\gamma}{2N}] \quad \text{(A2-59)}$$

$$\times \frac{r_1}{4\pi N} \exp[- \frac{r_1^{\,2}+\alpha_1^{\,2}(y_3)}{4N}] I_o[\frac{r_1\alpha_1(y_3)}{2N}]\} dr_o dr_1 dy_3$$

$$\Pr(\text{err } H_1) = \int_{r_1=0}^{\infty}\int_{y_3=0}^{2\pi} \frac{r_1}{4\pi N} \exp [- \frac{r_1^{\,2}+\alpha_1^{\,2}(y_3)}{4N}] I_o[\frac{r_1\alpha_1(y_3)}{2N}] \quad \text{(A2-60)}$$

$$\times \{ \int_{r_o=r_1}^{\infty} \frac{r_o}{2N} \exp [- \frac{r_o^{\,2}+\gamma^2}{4N}] I_o [\frac{r_o\gamma}{2N}]dr_o \} dr_1 dy_3$$

With the change of variable

$$w = r_o/ \sqrt{2N}$$

the bracketed $\{\cdot\}$ integration is

$$\int_{w_1=r_1/\sqrt{2N}}^{\infty} w \exp[-\frac{1}{2}(w^2 + (\gamma^2/2N))] \, I_o[w\gamma/\sqrt{2N}]dw \tag{A2-61}$$

$$= \quad Q[(\gamma/\sqrt{2N}), \ (r_1/\sqrt{2N})]$$

where

$$Q(\alpha,\beta) = \int_{\beta}^{\infty} t\exp[-\frac{1}{2}(t^2+\alpha^2)] \, I_o(\alpha t)dt \tag{A2-62}$$

is the tabulated Marcum Q-function [26], which also has the following properties:

$$Q(\alpha,0) = 1 \tag{A2-63}$$

$$Q(\alpha,\beta) = 1 + \exp[-\frac{1}{2}(\alpha^2+\beta^2)] \, I_o(\alpha\beta) - Q(\beta,\alpha) \tag{A2-64}$$

$$Q(0,\beta) = \exp\left[-\frac{1}{2}\beta^2\right] \tag{A2-65}$$

thus with the substitution of the Q-function and with the new definition $w = (r_1/\sqrt{2N})$, (A2-60) may be written

$$\Pr(\mathrm{err}|H_1) = \int_{y_3=0}^{2\pi} (1/2\pi)\{ \int_{w=0}^{\infty} w\exp[-\frac{1}{2}(w^2+(\alpha_1{}^2(y_3)/2N)] \quad \text{(A2-66)}$$

$$\times \; I_o[w\alpha_1(y_3)/\sqrt{2N}\,] \;\; Q[\gamma/\sqrt{2N},\, w]\; dw\}\; dy_3$$

Invoking the relation (A2-64) for the Q-function, the inner integral, which we denote as $g(y_3)$ is

$$g(y_3) = \int_{w=0}^{\infty} w\exp[-\frac{1}{2}(w^2+ (\alpha_1{}^2(y_3)/2N))]I_o[w\alpha_1(y_3)/\sqrt{2N}]\; dw \quad \text{(A2-67)}$$

$$+ \int_{w=0}^{\infty} w\exp[-\frac{1}{2}(w^2+(\alpha_1{}^2(y_3)/2N))]I_o[w\alpha_1(y_3)/\sqrt{2N}]$$

$$\times\; \exp[-\frac{1}{2}(w^2+(\gamma^2/2N))]I_o[w\gamma/\sqrt{2N}]dw$$

$$+ \int_{w=0}^{\infty} -\, w\exp[-\frac{1}{2}(w^2+(\alpha_1{}^2(y_3)/2N))]I_o[w\alpha_1(y_3)/\sqrt{2N}]$$

$$\times\;\; Q(w, \gamma/\sqrt{2N})dw$$

Using the property [25]

$$\int_o^{\infty} z\exp(-z^2)\exp[-\frac{1}{2}(\alpha^2+\beta^2)]I_o(\alpha z)I_o(\beta z)dz \quad \text{(A2-68)}$$

$$= \frac{1}{2}\exp[-\frac{1}{4}(\alpha^2+\beta^2)]\times I_o(\alpha\beta/2)$$

then the first integral is merely $Q(\alpha_1(y_3)/\sqrt{2N}, 0)$ via (A2-62) and has value unity via (A2-63), the second integral is expressible in the form of (A2-68) where

$$\alpha = \alpha_1(y_3)/\sqrt{2N}$$

$$\beta = \gamma/\sqrt{2N}$$

and the third integral is evaluated by substitution of the definition for $Q(w,\gamma/\sqrt{2N})$. We thus have:

$$g(y_3) = 1 + \frac{1}{2}\exp[-(\alpha_1{}^2(y_3)+\gamma^2)/8N] I_o[\gamma\alpha_1(y_3)/4N] \tag{A2-69}$$

$$+ \int_{w=0}^{\infty}\int_{t=\gamma/\sqrt{2N}}^{\infty} -w\exp[-\frac{1}{2}(w^2+(\alpha_1{}^2(y_3)/2N))]$$

$$\times I_o[w\alpha_1(y_3)/\sqrt{2N}] \cdot \{ t\exp[-\frac{1}{2}(t^2+w^2)] I_o[wt]\} dt dw$$

where $g(y_3)$ is the bracketed term in (A2-66).

Interchanging the order of integration and arranging terms yields:

$$g(y_3) = 1 + \frac{1}{2}\exp[-(\alpha_1{}^2(y_3)+\gamma^2)/8N] I_o[\gamma\alpha_1(y_3)/4N] \tag{A2-70}$$

$$+ \int_{t=\gamma/\sqrt{2N}}^{\infty} -t\{ \int_{w=0}^{\infty} w\exp(-w^2)\exp[-\frac{1}{2}(t^2+(\alpha_1{}^2(y_3)/2N))] \times$$

$$\times\ I_o[w\alpha_1(y_3)/\sqrt{2N}]I_o[wt]dw\}dt$$

the inner bracketed integral has the form of (A2-68) with the definitions

$$\alpha = \alpha_1(y_3)/\sqrt{2N}$$

$$\beta = t$$

thus:

$$g(y_3) = 1 + \frac{1}{2}\exp[-(\gamma^2+\alpha_1{}^2(y_3))/8N]I_o[\gamma\alpha_1(y_3)/4N] \tag{A2-71}$$

$$+ \int_{t=\gamma/\sqrt{2N}}^{\infty} -\frac{1}{2}\ t\exp[-\frac{1}{4}(t^2+(\alpha_1{}^2(y_3)/2N))]I_o[t\alpha_1(y_3)/2\sqrt{2N}]dt$$

with the change of variable $x = t/\sqrt{2}$, the remaining integral may be written as a Q-function. Thus we have for (A2-70):

$$g(y_3) = 1 + \frac{1}{2}\exp[-(\gamma^2+\alpha_1{}^2(y_3))/8N]I_o[\gamma\alpha_1(y_3)/4N] \tag{A2-72}$$

$$- Q[\alpha_1(y_3)/2\sqrt{N},\ \gamma/2\sqrt{N}]$$

using (A2-64) again for the above Q-function and collecting terms:

$$g(y_3) = Q[\gamma/2\sqrt{N}, \alpha_1(y_3)/2\sqrt{N}] - \frac{1}{2}\exp[-(\gamma^2+\alpha_1^{\,2}(y_3))/8N] \tag{A2-73}$$

$$\times\ I_o[\gamma\alpha_1(y_3)/4N]$$

thus from (A2-66) we have the error probability

$$\Pr(\text{err}|H_1) = (1/2\pi)\int_{y_3=0}^{2\pi} g(y_3)\,dy_3 \tag{A2-74}$$

using the definitions of γ^2 and $\alpha_1^{\,2}(y_3)$ as given in (A2-31) and (A2-54), we have:

$$\Pr(\text{err}|H_1) = \frac{1}{2\pi}\int_0^{2\pi}\{Q[\frac{(C^2+D^2)^{\frac{1}{2}}}{2\sqrt{N}}, \frac{(4A_s^{\,2}+B^2+C^2+4A_s(B\cos y_3-C\sin y_3))^{\frac{1}{2}}}{2\sqrt{N}}]$$

$$-\frac{1}{2}\exp[-(4A_s^{\,2}+B^2+2C^2+D^2+4A_s(B\cos y_3-C\sin y_3))/8N]$$

$$\times\ I_o[\ (4A_s^{\,2}+B^2+C^2+4A_s(B\cos y_3-C\sin y_3))^{\frac{1}{2}}(C^2+D^2)^{\frac{1}{2}}/4N]\}dy_3$$

$$\text{(A2-75)}$$

where B, C, D are as defined in equations (3-37) to (3-39). Equation (A2-75) is equivalent to (3-75) upon the substitution (3-61), thus the error relationships are identical.

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VITA

Cecil A. Lowery, Jr., was born and raised in Shreveport, Louisiana, the son of Cecil Sr. and Frances Lynn Lowery.

After graduation from Fair Park High School, he attended Louisiana State University at Baton Rouge, Louisiana, from which he obtained the B.S.E.E. degree *cum laude*. Upon joining the United Gas Research Laboratories in Shreveport, he worked in the Electronics Group designing data acquisition equipment. He returned to Louisiana State from which he obtained the M.S.E.E. degree in August 1969. Since that time he has attended the Georgia Institute of Technology full-time with the financial support of an NDEA Fellowship, and part-time since accepting a position in 1973 as Assistant Professor in the Electrical Engineering and Communication Sciences Department of Florida Technological University in Orlando, Florida, where he now resides.

He is a member of Phi Eta Sigma, Eta Kappa Nu, Tau Beta Pi, Phi Kappa Phi, and the Institute of Electrical and Electronic Engineers.